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Characterizations of Besov–Hardy–Sobolev Spaces: A Unified Approach

HANS TRIEBEL

Sektion Mathematik, Universität Jena, DDR-6900 Jena, German Democratic Republic

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1. INTRODUCTION AND DEFINITIONS

1.1. *Introduction*

This paper deals with the Besov–Hardy–Sobolev spaces $B^s_{p,q}(R_n)$ and $F^s_{p,q}(R_n)$ on the euclidean n-space R_n. It is a self-contained survey about a special aspect, the theory of equivalent norms and quasi-norms in these spaces. In [22] we dealt extensively with characterizations of the spaces considered via rather different means: differences $(\Delta^M_h f)(x)$ and derivatives $(D^\alpha f)(x)$ of functions, several types of mean values of differences of functions, traces of harmonic functions, and temperatures in $R^+_{n+1} = \{(x, t) \mid x \in R_n, t > 0\}$ on the hyperplane $t = 0$, etc. In that monograph we used specific tools for different characterizations. One aim of the present paper is to demonstrate that these different characterizations can be obtained from a unified point of view. In this sense this paper may be considered as the continuation and an improvement of [23] and of the Subsections 2.5.15–2.5.17 of the recent Russian edition of [22]. In order to make our presentations self-contained we include some arguments from these sources. The second aim of this paper is to extend the previous results. For that purpose we introduce weighted means of differences and derivatives of functions and distributions, which give also the possibility to characterize spaces $B^s_{p,q}(R_n)$ and $F^s_{p,q}(R_n)$ with negative smoothness s and to establish a localization principle for all these spaces in a rather easy way.

As in [22] we are mostly interested in the non-homogeneous spaces $B^s_{p,q}(R_n)$ and $F^s_{p,q}(R_n)$. But as a by-product of the presentation in this paper we also obtain corresponding characterizations of the homogeneous spaces $\dot{B}^s_{p,q}(R_n)$ and $\dot{F}^s_{p,q}(R_n)$ which are simpler, in general. In contrast to [22] we try to avoid the technique of maximal functions as far as possible (but not always). The reason is that we obtain on this way more natural restrictions for the parameters s, p, and q for the different types of characterizations.

The plan of the paper is as follows. In Subsection 1.2 we give the necessary definitions and add few further comments. Section 2 deals with

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general characterizations. The concrete examples which are an essential part of this paper are discussed in Section 3.

Although we try to make this paper essentially self-contained we shall not describe the historical background of the theory of the spaces $B^s_{p,q}(R_n)$ and $F^s_{p,q}(R_n)$ which may be found in the two books [20, 22]. But in Remark 10 and in 3.5 we give some more specific references as far as the topics treated in this paper are concerned.

As usual uninteresting positive constants are denoted by the same letter c, where their numerical values may differ from formula to formula.

1.2. *Definitions*

We follow essentially [22, in particular Subsect. 2.3.1]. Let R_n be the euclidean n-space. S and S' stand for the Schwartz space of all infinitely differentiable rapidly decreasing complex-valued functions on R_n and the collection of all complex-valued tempered distributions on R_n, resectively. Because all spaces in this paper are defined on R_n we omit "R_n" in the respective definitions. F and F^{-1} denote the Fourier transform and its inverse on S', respectively. Let $0<p\leqslant\infty$ then

$$\|f \mid L_p\| = \left(\int_{R_n} |f(x)|^p \, dx\right)^{1/p}$$

(usual modification if $p=\infty$). Let Φ be the collection of all systems $\{\varphi_j(x)\}_{j=0}^{\infty} \subset S$ with the following properties:

(i) $\varphi_j(x)=\varphi(2^{-j}x)$, if $j=1, 2, 3, \ldots$, (1)

(ii) $\operatorname{supp} \varphi_0 \subset \{x \mid |x| \leqslant 2\}$, $\operatorname{supp} \varphi \subset \{x \mid \frac{1}{2} \leqslant |x| \leqslant 2\}$, (2)

(iii) $\sum_{j=0}^{\infty} \varphi_j(x)=1$ for every $x \in R_n$. (3)

Remark 1. In contrast to [22, Subsect. 2.3.1] we use (1) from the very beginning. This is not a serious restriction, cf. [22, Remark 1 in 2.3.1]. If one uses more general systems $\{\varphi_j(x)\}_{j=0}^{\infty}$ then (2) must be replaced by

$$\operatorname{supp} \varphi_j \subset \{x \mid 2^{j-1} \leqslant |x| \leqslant 2^{j+1}\} \tag{4}$$

for $j=1, 2, 3, \ldots$ and one must be sure that for every multi-index α there exists a positive number c_α such that

$$2^{j|\alpha|}\, |D^\alpha \varphi_j(x)| \leqslant c_\alpha \qquad \text{for all } j=0, 1, 2, \ldots \text{ and all } x \in R_n. \tag{5}$$

In the above special case, (4) and (5) are satisfied.

Remark 2. Systems of the above type will be used in order to define the non-homogeneous spaces $B^s_{p,q}$ and $F^s_{p,q}$. For the definition of the homogeneous counterparts of the above spaces, Φ must be modified as follows. $\dot{\Phi}$ is the collection of all systems $\{\varphi_j(x)\}_{j=-\infty}^{\infty} \subset S$ with(1) for all integers j, (2), and (3) with $\sum_{j=-\infty}^{\infty}$ instead of $\sum_{j=0}^{\infty}$, for $x \in R_n - \{0\}$. Of course, Remark 1 can also be modified in an obvious way.

DEFINITION 1. Let $\{\varphi_j(x)\}_{j=0}^{\infty} \in \Phi$. Let $-\infty < s < \infty$ and $0 < q \leqslant \infty$.

(i) If $0 < p \leqslant \infty$ then

$$B_{p,q}^s = \left\{ f \mid f \in S', \|f \mid B_{p,q}^s\|^{\{\varphi_j\}} = \left(\sum_{j=0}^{\infty} 2^{sjq} \|F^{-1}\varphi_j Ff \mid L_p\|^q \right)^{1/q} < \infty \right\} \tag{6}$$

(usual modification if $q = \infty$).

(ii) If $0 < p < \infty$ then

$$F_{p,q}^s = \left\{ f \mid f \in S', \|f \mid F_{p,q}^s\|^{\{\varphi_j\}} = \left\| \left(\sum_{j=0}^{\infty} 2^{sjq} |(F^{-1}\varphi_j Ff)(\cdot)|^q \right)^{1/q} \Big| L_p \right\| < \infty \right\} \tag{7}$$

(usual modification if $q = \infty$).

Remark 3. This definition makes sense because by the Paley–Wiener–Schwartz theorem $(F^{-1}\varphi_j Ff)(x)$ is an analytic function for every $f \in S'$. For sake of simplicity we always write $(F^{-1}\varphi_j Ff)(x)$ instead of the more correct version $(F^{-1}[\varphi_j Ff])(x)$. In other words, F^{-1} is applied to the distribution $\varphi_j Ff$. The theory of these spaces has been developed in [22]. In particular, $B_{p,q}^s$ and $F_{p,q}^s$ are quasi-Banach spaces (Banach spaces if $p \geqslant 1$ and $q \geqslant 1$). In the sense of equivalent quasi-norms they are independent of the chosen system $\{\varphi_j\} \in \Phi$. This justifies to write $\|f \mid B_{p,q}^s\|$ and $\|f \mid F_{p,q}^s\|$ instead of $\|f \mid B_{p,q}^s\|^{\{\varphi_j\}}$ and $\|f \mid F_{p,q}^s\|^{\{\varphi_j\}}$, respectively, in the sequel, if $\{\varphi_j\}$ is an arbitrary system belonging to Φ. Furthermore these two scales cover many classical function spaces: Hölder spaces, Zygmund classes, Besov–Lipschitz spaces, Sobolev spaces, Bessel-potential spaces, and spaces of Hardy type. For details we refer to [22], but some of these claims are also essentially covered by the equivalent quasi-norms described in this paper.

Remark 4. In our terminology $B_{p,q}^s$ and $F_{p,q}^s$ are non-homogeneous spaces. We describe their homogeneous counterparts $\dot{B}_{p,q}^s$ and $\dot{F}_{p,q}^s$. For that purpose one replaces $\{\varphi_j(x)\}_{j=0}^{\infty} \in \Phi$ in Definition 1 by $\{\varphi_j(x)\}_{j=-\infty}^{\infty} \in \dot{\Phi}$ and $\sum_{j=0}^{\infty}$ in (6) and (7) by $\sum_{j=-\infty}^{\infty}$; cf. Remark 2. There is a small technical difficulty, because

$$\|f \mid \dot{B}_{p,q}^s\|^{\{\varphi_j\}} = \|f \mid \dot{F}_{p,q}^s\|^{\{\varphi_j\}} = 0$$

if f is a polynomial. Hence these spaces should be considered modulo polynomials. A more detailed consideration of this point may be found in [22, Chap. 5]. Again we write $\|f \mid \dot{B}^s_{p,q}\|$ and $\|f \mid \dot{F}^s_{p,q}\|$ instead of $\|f \mid \dot{B}^s_{p,q}\|^{\{\varphi_j\}}$ and $\|f \mid \dot{F}^s_{p,q}\|^{\{\varphi_j\}}$, respectively, in the sequel.

DEFINITION 2. Let $\{\varphi_j(x)\}_{j=0}^{\infty} \in \Phi$ or $\{\varphi_j(x)\}_{j=-\infty}^{\infty} \in \dot{\Phi}$ and let $f \in S'$. Let $a > 0$. Then

$$(\varphi_j^* f)_a(x) = \sup_{y \in R_n} \frac{|(F^{-1}\varphi_j Ff)(x-y)|}{1+|2^j y|^a}, \qquad x \in R_n, \tag{8}$$

(maximal function).

Remark 5. Occasionally we shall use the technique of maximal functions in this paper, sometimes also for more general systems $\{\varphi_j\}$. As a special case of the theorem in Subsection 2.3.6 in [22] we recall the following result.

(i) If $a > n/p$ and $\{\varphi_j(x)\}_{j=0}^{\infty} \in \Phi$ then

$$\left(\sum_{j=0}^{\infty} 2^{jsq} \, \|(\varphi_j^* f)_a \mid L_p\|^q \right)^{1/q} \tag{9}$$

(modification if $q = \infty$) is an equivalent quasi-norm on $B^s_{p,q}$.

(ii) If $a > n/\min(p, q)$ and $\{\varphi_j(x)\}_{j=0}^{\infty} \in \Phi$ then

$$\left\| \left(\sum_{j=0}^{\infty} 2^{jsq} \mid (\varphi_j^* f)_a(\cdot)|^q \right)^{1/q} \mid L_p \right\| \tag{10}$$

(modification if $q = \infty$) is an equivalent quasi-norm in $F^s_{p,q}$. There is a complete counterpart for the homogeneous spaces $\dot{B}^s_{p,q}$ and $\dot{F}^s_{p,q}$. Of course, (i) and (ii) are valid for all admissible values of s, p, and q in the sense of Definition 1.

2. GENERAL CHARACTERIZATIONS

2.1. *Spaces of Type $F^s_{p,q}$: Basic Results*

Let $0 < p \leqslant \infty$ and $0 < q \leqslant \infty$. We introduce the abbreviations

$$\sigma_p = n\left(\frac{1}{\min(p, 1)} - \frac{1}{2}\right) \quad \text{and} \quad \sigma_{p,q} = n\left(\frac{1}{\min(p, q, 1)} - \frac{1}{2}\right) \tag{11}$$

and

$$\tilde{\sigma}_p = n\left(\frac{1}{\min(p, 1)} - 1\right) \quad \text{and} \quad \tilde{\sigma}_{p,q} = n\left(\frac{1}{\min(p, q, 1)} - 1\right); \tag{12}$$

cf. [22, 2.5.3]. Let $h(x) \in S$ and $H(x) \in S$ with

$$\operatorname{supp} h \subset \{ y \mid |y| \leqslant 2 \}, \qquad \operatorname{supp} H \subset \{ y \mid \tfrac{1}{4} \leqslant |y| \leqslant 4 \},$$
$$h(x) = 1 \quad \text{if} \quad |x| \leqslant 1 \quad \text{and} \quad H(x) = 1 \quad \text{if} \quad \tfrac{1}{2} \leqslant |x| \leqslant 2.$$

THEOREM 1. *Let $0 < p < \infty$, $0 < q \leqslant \infty$, and $-\infty < s < \infty$. Let s_0 and s_1 be two real numbers with*

$$s_0 + \tilde{\sigma}_{p,q} < s < s_1 \tag{13}$$

and $s_1 > \tilde{\sigma}_p$. Let $\varphi_0(x)$ and $\varphi(x)$ be infinitely differentiable complex-valued functions on R_n and $R_n - \{0\}$, respectively, which satisfy the Tauberian conditions

$$|\varphi_0(x)| > 0 \qquad \text{if} \quad |x| \leqslant 2 \tag{14}$$

and

$$|\varphi(x)| > 0 \qquad \text{if} \quad \tfrac{1}{2} \leqslant |x| \leqslant 2. \tag{15}$$

Let $a > n/\min(p, q)$,

$$\int_{R_n} \left| \left(F^{-1} \frac{\varphi(z)\, h(z)}{|z|^{s_1}} \right)(y) \right| (1 + |y|)^a \, dy < \infty, \tag{16}$$

$$\sup_{m = 1,2,\dots} 2^{-ms_0} \int_{R_n} |(F^{-1} \varphi(2^m \cdot)\, H(\cdot))(y)| \, (1 + |y|)^a \, dy < \infty, \tag{17}$$

and

$$\sup_{m = 1,2,\dots} 2^{-ms_0} \int_{R_n} |(F^{-1} \varphi_0(2^m \cdot)\, H(\cdot))(y)| \, (1 + |y|)^a \, dy < \infty. \tag{18}$$

Let

$$\varphi_j(x) = \varphi(2^{-j}x) \qquad \text{if} \quad x \in R_n - \{0\} \tag{19}$$

and $j = 1, 2, , \dots$. Then

$$\left\| \left(\sum_{j=0}^{\infty} 2^{jsq} \, |(F^{-1} \varphi_j F f)(\cdot)|^q \right)^{1/q} \Big| L_p \right\| \tag{20}$$

(*modification if $q = \infty$*) *is an equivalent quasi-norm in $F^s_{p,q}$.*

Proof. *Step* 1. In the first two steps we prove that the quasi-norm in (20) can be estimated from above by $c\,\|f \mid F^s_{p,q}\|$. Let $\{\rho_j(x)\}_{j=0}^{\infty} \in \Phi$, where we replaced the φ_j's in the original definition by the ρ_j's. In particular we have $\rho_j(x)=\rho(2^{-j}x)$ if $j=1, 2, 3, \ldots$. Let $\rho_j(x)=0$ if $j=-1, -2, \ldots$. Then we obtain

$$2^{js}(F^{-1}\varphi_j Ff)(x)=2^{js}\sum_{l=-\infty}^{\infty}(F^{-1}\varphi_j\rho_{l+j}Ff)(x)=\sum_{l=-\infty}^{K}\cdots+\sum_{l=K+1}^{\infty}\cdots, \qquad (21)$$

where K is a natural number which will be chosen later on. We estimate the first sum. Let

$$\tilde{\rho}_0(x)=|x|^{s_1}\rho_0(x), \qquad \tilde{\rho}(x)=|x|^{s_1}\rho(x), \qquad \text{and}$$
$$\tilde{\rho}_j(x)=\tilde{\rho}(2^{-j}x) \qquad \text{if} \quad j=1, 2, \ldots. \qquad (22)$$

Then we have

$$\left|\sum_{l=-\infty}^{K}2^{js}(F^{-1}\varphi_j\rho_{l+j}Ff)(x)\right|$$
$$\leqslant \sum_{l=-\infty}^{K}2^{l(s_1-s)}\left|\left(F^{-1}\frac{\varphi_j(z)}{|2^{-j}z|^{s_1}}2^{s(j+l)}\tilde{\rho}_{j+l}(z)\,Ff\right)(x)\right|. \qquad (23)$$

One can replace $\varphi_j(z)$ on the right-hand side of (23) by $\varphi_j(z)\,h(c2^{-j}z)$, where c is an appropriate positive number which depends on K. Let $j=1, 2, \ldots$. Recall $\varphi_j(z)=\varphi(2^{-j}z)$. Then $|(F^{-1}\cdots)(x)|$ on the right-hand side of (23) can be estimated from above by

$$\int_{R_n}\left|\left(F^{-1}\frac{\varphi(2^{-j}z)\,h(c2^{-j}z)}{|2^{-j}z|^{s_1}}\right)(y)\right|\,|(F^{-1}2^{s(j+l)}\tilde{\rho}_{j+l}Ff)(x-y)|\,dy. \qquad (24)$$

Recall $(F^{-1}\lambda(2^{-j}\cdot))(y)=2^{jn}(F^{-1}\lambda)(2^jy)$. We apply this formula to the first factor in (24) and replace afterwards 2^jy by y. We use (8) with $\tilde{\rho}_j$ instead of φ_j. Then we have

$$|(F^{-1}2^{s(j+l)}\tilde{\rho}_{j+l}Ff)(x-2^{-j}y)|\leqslant c2^{(j+l)s}(\tilde{\rho}^*_{j+l}f)_a(x)(1+|y|)^a, \qquad (25)$$

where c depends on K (but not on x, y, j, and l). We put these transformations and estimates in (23) and obtain

$$\left|\sum_{l=-\infty}^{K}2^{js}(F^{-1}\varphi_j\rho_{l+j}Ff)(x)\right|\leqslant c\sum_{l=-\infty}^{K}2^{l(s_1-s)}2^{(j+l)s}(\tilde{\rho}^*_{j+l}f)_a(x), \qquad (26)$$

where $j=1, 2, 3, \ldots$. We used (16). We apply first the l_q-quasi-norm with respect to j and afterwards the L_p-quasi-norm with respect to x. Because $s_1 > s$ we obtain

$$\left\| \left(\sum_{j=1}^{\infty} \left| \sum_{l=-\infty}^{K} 2^{js}(F^{-1}\varphi_j \rho_{l+j} Ff)(\cdot) \right|^q \right)^{1/q} \,\Big|\, L_p \right\| \leqslant c \left\| \left(\sum_{m=0}^{\infty} 2^{msq} (\tilde{\rho}_m^* f)_a^q(\cdot) \right)^{1/q} \,\Big|\, L_p \right\|. \tag{27}$$

Because $a > n/\min(p, q)$ we can use the vector-valued maximal inequality from [22, Subsect. 1.6.2], only the term with $\tilde{\rho}_0(x) = |x|^{s_1} \rho_0(x)$ is critical. We return to this point later on in Remark 6 where we prove that

$$\|F^{-1}\tilde{\rho}_0 Ff \mid L_p\| \leqslant c \, \|F^{-1}\rho_0 Ff \mid L_p\| \tag{28}$$

holds provided that $s_1 > \tilde{\sigma}_p$. Then the maximal inequality can be applied and (27) can be estimated from above by

$$c \left\| \left(\sum_{m=0}^{\infty} 2^{msq} \, |(F^{-1}\tilde{\rho}_m Ff)(\cdot)|^q \right)^{1/q} \,\Big|\, L_p \right\|. \tag{29}$$

Finally, by (28) and the vector-valued Fourier multiplier theorem from [22, 1.6.3], (29), and consequently (27), can be estimated from above by

$$c \left\| \left(\sum_{m=0}^{\infty} 2^{msq} \, |(F^{-1}\rho_m Ff)(\cdot)|^q \right)^{1/q} \,\Big|\, L_p \right\| = c \, \|f \mid F_{p,q}^s\|. \tag{30}$$

The term with $j = 0$ can be incorporated afterwards. In other words, we have

$$\left\| \left(\sum_{j=0}^{\infty} \left| \sum_{l=-\infty}^{K} 2^{js}(F^{-1}\varphi_j \rho_{l+j} Ff)(\cdot) \right|^q \right)^{1/q} \,\Big|\, L_p \right\| \leqslant c \, \|f \mid F_{p,q}^s\|, \tag{31}$$

where c depends on K.

Step 2. We estimate the second sum in (21), where we now calculate carefully the dependence of the constants on K. The counterpart of (22) reads as follows,

$$\hat{\rho}(x) = |x|^{s_0} \rho(x), \qquad \hat{\rho}_j(x) = \hat{\rho}(2^{-j}x) \qquad \text{if} \quad j = 1, 2, \dots. \tag{32}$$

Instead of (23) we have now

$$\left| \sum_{l=K+1}^{\infty} 2^{js}(F^{-1}\varphi_j \rho_{l+j} Ff)(x) \right| \leqslant \sum_{l=K+1}^{\infty} 2^{l(s_0-s)} \left| \left(F^{-1} \frac{\varphi_j(z)}{|2^{-j}z|^{s_0}} H(2^{-j-l}z)\, 2^{s(j+l)} \hat{\rho}_{j+l}(z)\, Ff \right)(x) \right|. \tag{33}$$

We have obvious counterparts of (24) and (25), where we replace $\tilde{\rho}$ and $x-2^{-j}y$ in (25) by $\hat{\rho}$ and $x-2^{-j-l}y$, respectively. Then we obtain

$$\left| \sum_{l=K+1}^{\infty} 2^{js}(F^{-1}\varphi_j \rho_{l+j} Ff)(x) \right| \leqslant c \sum_{l=K+1}^{\infty} 2^{l(s_0-s)} 2^{(j+l)s} (\hat{\rho}^*_{j+l} f)_a(x), \tag{34}$$

where we used (17), (18),

$$\int_{R_n} \left| \left(F^{-1} \frac{\varphi(2^m z)\, H(z)}{|2^m z|^{s_0}} \right)(y) \right| (1+|y|^a)\, dy \leqslant c 2^{-ms_0} \int_{R_n} |(F^{-1}\varphi(2^m z)\, H(z))(y)|\, (1+|y|^a)\, dy \tag{35}$$

and a similar estimate with φ_0 instead of φ. Estimate (34) is the counterpart of (26) where c is independent of K. In the same way as in the first step we arrive at

$$\left\| \left(\sum_{j=0}^{\infty} \left| \sum_{l=K+1}^{\infty} 2^{js}(F^{-1}\varphi_j \rho_{l+j} Ff)(\cdot) \right|^q \right)^{1/q} \Big| L_p \right\| \leqslant c 2^{-K(s-s_0)} \|f \mid F^s_{p,q}\|, \tag{36}$$

where we used $s_0 < s$. The constant c in (36) is independent of K. Now, (21), (31), and (36) prove that the quasi-norm in (20) can be estimated from above by $c\, \|f \mid F^s_{p,q}\|$.

Step 3. We prove that $\|f \mid F^s_{p,q}\|^{\{\rho_m\}}$ can be estimated from above by the quasi-norm in (20), where $\{\rho_m(x)\}_{m=0}^{\infty} \in \Phi$ has the above meaning. Let $\psi(x) \in S$ be a function with $\operatorname{supp} \psi \subset \{y \mid |y| \leqslant 2^{K+1}\}$ and $\psi(x)=1$ if $|x| \leqslant 2^K$, where we choose the natural number K later on. By (14), (15), (19), and the properties of the functions ρ_j, we have

$$|(F^{-1}\rho_j Ff)(x)| = |(F^{-1}\rho_j \psi(2^{-j}\cdot)\, Ff)(x)| \leqslant c \int_{R_n} \left| \left(F^{-1} \frac{\rho_j}{\varphi_j} \right)(y) (F^{-1}\varphi_j \psi(2^{-j}\cdot)\, Ff)(x-y) \right| dy. \tag{37}$$

For fixed $x \in R_n$ the Fourier transform of the y-function in the integral in (37) has a support contained in a ball of radius $c2^{j+K}$, where c is independent of j and K. Let $0<r<\min(1, p, q)$. We use an inequality of Nikol'skij type, cf. [22, (1.3.2/5)], and obtain

$$|(F^{-1}\rho_j Ff)(x)|^r \leqslant c2^{(j+K)n(1-r)} \int_{R_n} \left| \left(F^{-1}\frac{\rho_j}{\varphi_j}\right)(y)(F^{-1}\varphi_j\psi(2^{-j}\cdot)\,Ff)(x-y)\right|^r dy. \quad (38)$$

Let $j=1, 2, 3, \ldots$. Then we have

$$\left|\left(F^{-1}\frac{\rho_j}{\varphi_j}\right)(y)\right|^r = 2^{jnr}\left|\left(F^{-1}\frac{\rho}{\varphi}\right)(2^jy)\right|^r \leqslant c2^{jnr}(1+|2^jy|)^{-b}, \quad (39)$$

where $b>0$ is at our disposal. A corresponding estimate holds for $j=0$. We put (39) in (38) and obtain

$$|(F^{-1}\rho_j Ff)(x)|^r \leqslant c2^{(j+K)n(1-r)+jnr} \sum_{l=0}^{\infty} 2^{-ld} \int_{\{y \mid |y| \leqslant 2^{-j+l}\}} |(F^{-1}\varphi_j\psi(2^{-j}\cdot)\,Ff)(x-y)|^r\, dy, \quad (40)$$

where $d>0$ is at our disposal. The integrals in (40) can be estimated from above by

$$c2^{-jn+ln}(M\,|F^{-1}\varphi_j\psi(2^{-j}\cdot)\,Ff|^r)(x), \quad (41)$$

where M stands for the Hardy–Littlewood maximal function. We put (41) in (40), choose $d>n$ and arrive at

$$|(F^{-1}\rho_j Ff)(x)|^r \leqslant c2^{Kn(1-r)}(M\,|F^{-1}\varphi_j\psi(2^{-j}\cdot)\,Ff|^r)(x). \quad (42)$$

Recall $1<p/r<\infty$ and $1<q/r\leqslant\infty$. We multiply both sides of (42) with 2^{jsr} and apply the $l_{q/r}$-norm with respect to j and afterwards the $L_{p/r}$-norm with respect to x. By the vector-valued maximal inequality due to Fefferman and Stein (cf. [7] or [22, 1.2.3]), which holds also for $q=\infty$, we obtain

$$\begin{aligned}
&\left\|\left(\sum_{j=0}^{\infty} |2^{js}(F^{-1}\rho_j Ff)(\cdot)|^q\right)^{1/q} \middle| L_p\right\|^r \\
&\quad\leqslant c2^{Kn(1-r)}\left\|\left(\sum_{j=0}^{\infty} (M\,|F^{-1}2^{js}\varphi_j\psi(2^{-j}\cdot)\,Ff|^r(\cdot))^{q/r}\right)^{r/q} \middle| L_{p/r}\right\| \\
&\quad\leqslant c'2^{Kn(1-r)}\left\|\left(\sum_{j=0}^{\infty} |2^{js}(F^{-1}\varphi_j\psi(2^{-j}\cdot)\,Ff)(\cdot)|^q\right)^{1/q} \middle| L_p\right\|^r \quad (43)
\end{aligned}$$

where c and c' are independent of K. Because

$$\varphi_j \psi(2^{-j}\cdot) = \varphi_j - \varphi_j(1 - \psi(2^{-j}\cdot))$$

the right-hand side of (43) can be estimated from above by the rth power of the quasi-norm in (20) (this is just what we want) and an additional term

$$c2^{Kn(1-r)} \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} \left| (F^{-1}\varphi_j(1-\psi(2^{-j}\cdot))\, Ff)(\cdot) \right|^q \right)^{1/q} \,\Big|\, L_p \right\|^r. \quad (44)$$

However, this term can be treated in the same way as in the second step, in particular we have an obvious counterpart of the estimate in (36). Hence, the term in (44) can be estimated from above by

$$c2^{Kr(n((1/r)-1)-(s-s_0))} \, \| f \mid F^s_{p,q} \|^r. \quad (45)$$

By (13), we may assume that $n((1/r)-1)-(s-s_0)<0$. Recall that the natural number K is at our disposal. We choose K large. Then the term in (45) can be estimated from above, say, by $\frac{1}{2} \| f \mid F^s_{p,q} \|^r$. Now (43) and the above splitting yield the desired estimate.

Remark 6. We prove the estimate in (28). This problem can be reduced to

$$\| F^{-1} |x|^{s_1} \sigma(x)\, Fg \mid L_p \| \leqslant c \, \| g \mid L_p \|, \quad (46)$$

where $\sigma(x) \in S$ with $\sigma(x)=1$ if $|x| \leqslant 1$ and $\sigma(x)=0$ if $|x| \geqslant 2$, and $g \in L_p$ is an arbitrary function with $\operatorname{supp} Fg \subset \{ y \mid |y| \leqq 1 \}$. By Theorem 1.5.2 in [22] this estimate is valid if

$$|x|^{s_1} \sigma(x) \in H^\kappa_2 \qquad \text{with} \quad \kappa > \sigma_p, \quad (47)$$

cf. (11), where H^κ_2 are the usual Bessel-potential spaces on R_n (Sobolev spaces if κ is a natural number). Because $s_1 > \tilde{\sigma}_p$ we may assume that $s_1 + (n/2) > \kappa > \sigma_p$. Let $\lambda(x)$ be an infinitely differentiable function on R_n with $\lambda(x)=1$ if $|x| \geqslant 2$ and $\lambda(x)=0$ if $|x| \leqslant 1$. Then $\{ |x|^{s_1} \sigma(x)\, \lambda(2^j x) \}_{j=1}^{\infty}$ is a fundamental sequence in H^κ_2. This follows by straightforward calculations if κ is a natural number. For fractional numbers κ it is a matter of interpolation or of the so-called multiplicative inequalities for Bessel-potential spaces. This completes the proof of (28). We add a further observation which will be useful later on. In Steps 1 and 2 we used only $s_0 < s$. In other words, in order to estimate the quasi-norms in (20) from above by $\| f \mid F^s_{p,q} \|$ the condition $s > s_0$ is sufficient.

In the following corollary we extend the notation $(\varphi_j^* f)_a(x)$ from (8) to the functions φ_j under consideration here.

COROLLARY 1. *Let* $0<p<\infty$, $0<q\leqslant\infty$, *and* $-\infty<s<\infty$. *Let* $a>n/\min(p,q)$. *Let* s_0 *and* s_1 *be two real numbers with*

$$s_0+a<s<s_1 \tag{48}$$

and $s_1>\tilde{\sigma}_p$. *Let* $\varphi_0(x)$ *and* $\varphi(x)$ *be the two functions from Theorem* 1 *with* (14)–(18). *Let* $\varphi_j(x)$ *with* $j=1,2,3,\dots$ *be given by* (19). *Then*

$$\left\| \left(\sum_{j=0}^{\infty} 2^{jsq} (\varphi_j^* f)_a(\cdot)^q \right)^{1/q} \middle| L_p \right\| \tag{49}$$

(*modification if* $q=\infty$) *is an equivalent quasi-norm in* $F^s_{p,q}$.

Proof. One can follow the proof of Theorem 1. We indicate the necessary modifications. We begin with (21) with $x-2^{-j}z$ instead of x. Then we use (25) with $x-2^{-j}y-2^{-j}z$ instead of $x-2^{-j}y$ and the additional factor $(1+|z|)^a$ on the right-hand side. We divide both sides of the modified estimate (26) by $(1+|z|)^a$ and take afterwards the supremum with respect to $z\in R_n$. This yields (31) with

$$\sup_{z\in R_n} \frac{|(F^{-1}\varphi_j\rho_{l+j}Ff)(x-2^{-j}z)|}{(1+|z|)^a} \qquad \text{instead of } (F^{-1}\varphi_j\rho_{l+j}Ff)(x). \tag{50}$$

We modify the second step of the proof of Theorem 1 in the same way. Then we obtain (34) with $x-2^{-j}z$ on the left-hand side and the additional factor $2^{la}(1+|z|)^a$ on the right-hand side. Because now $s_0+a<s$ we obtain (36) with the same substitute as in (50). This proves that the quasi-norm in (49) can be estimated from above by $\|f\mid F^s_{p,q}\|$. The other direction follows from Theorem 1 because $\tilde{\sigma}_{p,q}<a$.

Remark 7. In this paper we are not so much interested in equivalent quasi-norms where maximal functions are involved. We included the above corollary for sake of completeness and in order to emphasize the difference between (13) and (48) which will be of crucial importance later on.

There is an immediate counterpart both of Theorem 1 and Corollary 1 for the homogeneous spaces $\dot{F}^s_{p,q}$. The proofs are essentially the same, but simpler. We formulate the result.

COROLLARY 2. *Let* $0<p<\infty$, $0<q\leqslant\infty$, *and* $-\infty<s<\infty$. *Let* $a>n/\min(p,q)$. *Let* $\varphi(x)$ *be an infinitely differentiable complex-valued function on* $R_n-\{0\}$ *which satisfies* (15), *and let* $\varphi_j(x)$ *be given by* (19), *where* j *is an arbitrary integer.*

(i) *Let* s_0 *and* s_1 *be two real numbers with* (13). *Let* (16) *and* (17) *be satisfied. Then*

$$\left\| \left(\sum_{j=-\infty}^{\infty} 2^{jsq} \, |(F^{-1}\varphi_j Ff)(\cdot)|^q \right)^{1/q} \Big| \, L_p \right\| \tag{51}$$

(*modification if* $q=\infty$) *is an equivalent quasi-norm in* $\dot{F}^s_{p,q}$.

(ii) *Let* s_0 *and* s_1 *be two real numbers with* (48). *Let* (16) *and* (17) *be satisfied. Then*

$$\left\| \left(\sum_{j=-\infty}^{\infty} 2^{jsq} (\varphi_j^* f)_a(\cdot)^q \right)^{1/q} \Big| \, L_p \right\| \tag{52}$$

(*modification if* $q=\infty$) *is an equivalent quasi-norm in* $\dot{F}^s_{p,q}$.

Remark 8. As remarked above the proof is the same as the proofs of Theorem 1 and Corollary 1. The assumption $s_1 > \tilde{\sigma}_p$ is now not necessary and, of course, also not the assumptions for $\varphi_0(x)$.

Our considerations in Section 3 are mainly based on conditions of type (16)–(18) both for the spaces $F^s_{p,q}$ and $B^s_{p,q}$. However, it will be useful to give a more handsome reformulation of these conditions. Recall that H^σ_2 are the Bessel-potential spaces on R_n (Sobolev spaces if σ is a natural number).

COROLLARY 3. *Let the hypotheses of Theorem* 1 *for the numbers* p, q, s, s_0, s_1 *and* a *be satisfied. Let* $\sigma > n/2 + a$. *Let* $\varphi_j(x)$ *be the functions from Theorem* 1 *where* (16)–(18) *are replaced by*

$$\left\| \frac{\varphi(x)\, h(x)}{|x|^{s_1}} \Big| \, H^\sigma_2 \right\| < \infty, \tag{16'}$$

$$\sup_{m=1,2,\ldots} 2^{-ms_0} \|\varphi(2^m \cdot)\, H(\cdot) \mid H^\sigma_2\| < \infty, \tag{17'}$$

and

$$\sup_{m=1,2,\ldots} 2^{-ms_0} \|(\varphi_0(2^m \cdot)\, H(\cdot) \mid H^\sigma_2\| < \infty, \tag{18'}$$

respectively. Then (20) *is an equivalent quasi-norm in* $F^s_{p,q}$.

Proof. Let $0 < r \leqslant 1$ *and* $-\infty < b < \infty$. Then

$$\|(F^{-1}\lambda)(y)(1+|y|)^b \mid L_r\| \leqslant c\, \|\lambda \mid H^\delta_2\| \tag{53}$$

with $\delta > n((1/r)-(1/2))+b$. This is a well-known estimate; cf. e.g., [21, p. 60] or [19, 22] (the proof in [22, 1.5.2] for $b=0$ can also be extended immediately to $b \neq 0$). However, (53) with $r=1$ and $a=b$ shows that (16)–(18) follow from (16′)–(18′), respectively.

Remark 9. In particular if σ is a natural number then the conditions (16′)–(18′) can be checked easily. Later on we shall use a combination of the conditions (16)–(18) and (16′)–(18′). Of course the conditions in the Corollaries 1 and 2 can also be re-formulated in the sense of the above corollary.

Remark 10. Considerations of the above type are not new, both for the spaces $F^s_{p,q}$ and $B^s_{p,q}$, and also for their homogeneous counterparts (cf. [22, 2.3.6, in particular Remark 3] where we have given some references to preceding papers). However, as far as conditions of type (13) or (48) (and their even better counterparts for the spaces $B^s_{p,q}$ which will be described in Subsect. 2.3) are concerned, the corresponding assertions in [22, Subsect. 2.3.6] are very rough and only of restricted value in order to obtain equivalent quasi-norms. Furthermore the great service of the Tauberian conditions (14), (15) was simply overlooked in [22]. A first improvement was obtained in [23], mostly for the spaces $B^s_{p,q}$. We included some of this material in the Subsections 2.5.15–2.5.17 of the recent Russian edition of [22]. Tauberian conditions have a long history. As far as the systematic use of ideas of the above type in the framework of the theory of function spaces and related problems in approximation theory are concerned we refer to Shapiro [17, 18]. Furthermore, Riviere and Madych developed in [16, 10] this method in great detail in order to study Hölder spaces. Some results in this connection for the spaces $\dot{B}^s_{p,\infty}$ with $1 \leqslant p \leqslant \infty$ may also be found in [15, Chap. 8] (cf. also [1] for further references and useful informations). As far as a unified approach to the theory of equivalent quasi-norms in the spaces $B^s_{p,q}$ and $F^s_{p,q}$ is concerned the results of this paper seem to be new, in particular if $p<1$.

2.2. *Spaces of Type $F^s_{p,q}$: Modifications*

Let $\varphi(x)=(e^{i\gamma x}-1)^M$, where γx stands for the scalar product of the variable $x \in R_n$ and $\gamma \in R_n$, and M is a natural number. Then $(F^{-1}\varphi Ff)(x)=(\Delta^M_\gamma f)(x)$, where Δ^M_γ are the usual differences of functions on R_n. Characterizations of spaces of type $F^s_{p,q}$ and $B^s_{p,q}$ via differences Δ^M_γ are very desirable. For appropriate numbers s_0 and s_1 hypotheses of type (16) and (17) for the above function $\varphi(x)$ are fulfilled, but not the Tauberian condition (15). On the other hand if one does not deal with a single function $(e^{i\gamma x}-1)^M$, but with an appropriate finite set of these functions $\{(e^{i\gamma^k x}-1)^M\}_{k=1}^N$ or families of these functions of type $\{(e^{i\gamma x}-1)^M\}_{|\gamma|=1}$ or $\{(e^{i\gamma x}-1)^M\}_{1\leqslant|\gamma|\leqslant 2}$ then the Tauberian condition (15) is satisfied in some sense and one can expect characterizations of type (20). In this subsection we describe the necessary modifications, compared with the preceding subsection. We are not interested in most general formulations but we restrict our attention just to those functions φ which cover the examples which we have in mind. The numbers σ_p, etc., and the functions $h(x)$ and $H(x)$ have the same meaning as at the beginning of Subsection 2.1. Furthermore, R_1 stands for the real line.

THEOREM 2. *Let $0<p<\infty$, $0<q\leqslant\infty$, and $-\infty<s<\infty$. Let s_0 and s_1 be two real numbers with (13) and $s_1>\tilde{\sigma}_p$. Let $\varphi_0(x)$ be the same function as in Theorem 1, including (14) and (18), where $a>n/\min(p,q)$. Let $\varphi(t)$ be an infinitely differentiable complex-valued function on $R_1-\{0\}$ which satisfies*

$$|\varphi(t)|>0 \qquad \text{if } \tfrac{1}{8}<t<8 \tag{54}$$

(Tauberian condition) and

$$\sup_{1\leqslant|\gamma|\leqslant 2}\int_{R_n}\left|\left(F^{-1}\frac{\varphi(\gamma z)\,h(z)}{|z|^{s_1}}\right)(y)\right|(1+|y|)^a\,dy<\infty, \tag{55}$$

$$\sup_{1\leqslant|\gamma|\leqslant 2}\ \sup_{m=1,2,\ldots} 2^{-ms_0}\int_{R_n}|(F^{-1}\varphi(2^m\gamma z)\,H(z))(y)|\,(1+|y|)^a\,dy<\infty. \tag{56}$$

Then

$$\|F^{-1}\varphi_0 Ff\mid L_p\|+\left\|\left(\int_{|h|\leqslant 1}|h|^{-sq}\,|(F^{-1}\varphi(h\cdot)\,Ff)(\cdot)|^q\frac{dh}{|h|^n}\right)^{1/q}\,\Big|\,L_p\right\| \tag{57}$$

and

$$\|F^{-1}\varphi_0 Ff\mid L_p\|+\left\|\left(\int_0^1 t^{-sq}\sup_{t/2\leqslant|h|\leqslant t}|(F^{-1}\varphi(h\cdot)\,Ff)(\cdot)|^q\frac{dt}{t}\right)^{1/q}\,\Big|\,L_p\right\| \tag{58}$$

(modification if $q=\infty$) are equivalent quasi-norms in $F^s_{p,q}$.

Proof. *Step* 1. We use the same argument as in Step 1 of the proof of Theorem 1 with $\varphi(2^{-j}\gamma x)$ instead of $\varphi_j(x)$ if $j=1,2,\ldots$ and $1\leqslant|\gamma|\leqslant 2$. Then we obtain (26). We take the supremum with respect to these γ's and proceed afterwards in the same way as in (27)–(31). We obtain

$$\begin{aligned}\left\|\left(\sum_{j=1}^{\infty}\left(\sum_{l=-\infty}^{K}2^{js}\sup_{1\leqslant|\gamma|\leqslant 2}|(F^{-1}\varphi(2^{-j}\gamma z)\,\rho_{l+j}(z)\,Ff)(\cdot)|\right)^q\right)^{1/q}\,\Big|\,L_p\right\|&\\ \leqslant c\,\|f\mid F^s_{p,q}\|&\end{aligned} \tag{59}$$

and a corresponding estimate with φ_0. Next we use the same arguments as in Step 2 of the proof of Theorem 1. We arrive at (34) with $\varphi(2^{-j}\gamma x)$

instead of $\varphi_j(x)$, where $j=1,2,\ldots$ and $1\leqslant|\gamma|\leqslant 2$. We take the supremum with respect to these γ's and obtain the following counterpart of (36),

$$\left\|\left(\sum_{j=1}^{\infty}\left(\sum_{l=K+1}^{\infty}2^{js}\sup_{1\leqslant|\gamma|\leqslant 2}|(F^{-1}\varphi(2^{-j}\gamma z)\,\rho_{l+j}(z)\,Ff)(\cdot)|\right)^{q}\right)^{1/q}\middle|\,L_p\right\| \leqslant c2^{-K(s-s_0)}\,\|f\mid F^s_{p,q}\| \tag{60}$$

and a corresponding estimate with φ_0. However, the quasi-norm in (57) can be estimated from above by the quasi-norm in (58), which in turn can be estimated from above by the sum of the left-hand sides of (59) and (60) and corresponding terms with φ_0, and hence by $\|f\mid F^s_{p,q}\|$.

Step 2. In order to prove the reverse inequality we must modify Step 3 of the proof of Theorem 1. Let γ with $1\leqslant|\gamma|\leqslant 2$ be given. Then $\{x\mid \frac{1}{2}\leqslant|x|\leqslant 2,\ |\varphi(\gamma x)|>0\}$ covers a sectorial set $\Omega_\gamma=\{y\mid\frac{1}{2}\leqslant|y|\leqslant 2, |(y/|y|)-(\gamma/|\gamma|)|\leqslant b\}$ for some $b>0$. Let again $\{\rho_m(x)\}_{m=0}^{\infty}\in\Phi$ with $\rho_m(x)=\rho(2^{-m}x)$ if $m=1,2,\ldots$. We decompose the basic function $\rho(x)$ by $\rho(x)=\sum_{k=1}^{N}\rho^{(k)}(x)$ such that for any γ with $1\leqslant|\gamma|\leqslant 2$ we find a number k such that $\operatorname{supp}\rho^{(k)}\subset\Omega_\gamma$. Thus is always possible if one chooses N large enough. Let k be given and let $1\leqslant|\gamma|\leqslant 2$ with $\operatorname{supp}\rho^{(k)}\subset\Omega_\gamma$. We follow the arguments of Step 3 of the proof of Theorem 1 with $\rho^{(k)}(2^{-j}x)$ and $\varphi(2^{-j}\gamma x)$ instead of ρ_j and φ_j, respectively. We arrive at the counterparts of (38) and (39), where in the latter inequality we may assume that the corresponding right-hand side is independent of k and γ. We substitute this inequality in the just-described modification of (38) and integrate over those γ's which are connected with the given number k in the above sense. Afterwards this integration can be extended to all admissible γ's. Then the corresponding right-hand sides are independent of k. Summation over k yields (40) with

$$\left(\int_{1\leqslant|\gamma|\leqslant 2}|(F^{-1}\varphi(2^{-j}\gamma z)\,\psi(2^{-j}z)\,Ff)(x-y)|^q\,d\gamma\right)^{1/q}$$

instead of $|(F^{-1}\varphi_j\psi(2^{-j}\cdot)\,Ff)(x-y)|$, where we used that

$$\int_\gamma|\cdots|^r\,d\gamma\leqslant c\left(\int_\gamma|\cdots|^q\,d\gamma\right)^{r/q}.$$

We have the counterparts of (43) and (44). We use the same splitting as after (43). The substitute of $|F^{-1}\varphi_j Ff|$ is $(\int_{1\leqslant|\gamma|\leqslant 2}|F^{-1}\varphi(2^{-j}\gamma\cdot)\,Ff|^q\,d\gamma)^{1/q}$ and the corresponding term yields (57). The remaining term, i.e., the counterpart of (44), can be estimated with the help of (60) in the same way as after (44). This shows that $\|f\mid F^s_{p,q}\|$ can be estimated from above by the quasi-norm in (57) and, consequently, also by the quasi-norm in (58). The proof is complete.

Remark 11. Under our specific assumptions for $\varphi(x)$ one can omit $\sup_{1\leqslant|\gamma|\leqslant 2}$ in (55) and (56), because it is sufficient to know corresponding estimates for a fixed $\gamma\neq 0$. However, if one replaces $\varphi(\gamma x)$ by more general families of functions $\varphi_\gamma(x)$, then one needs formulations of type (55) and (56). As we observed at the end of Remark 6 the assumption $s>s_0$ is sufficient in order to estimate the quasi-norms in (57), (58) (and also in (61) below if $s>0$) from above by the $F^s_{p,q}$-quasi-norm. Furthermore the Corollaries 1 and 3 have respective counterparts. The corresponding counterpart of Corollary 2 will be formulated separately later on.

COROLLARY 4. *Let all the hypotheses of Theorem 2 be fulfilled and let in addition $s>0$. Then*

$$\|F^{-1}\varphi_0 Ff \mid L_p\| + \left\| \left(\int_0^1 t^{-sq} \sup_{0<|h|\leqslant t} |(F^{-1}\varphi(h\cdot)\, Ff)(\cdot)|^q \frac{dt}{t}\right)^{1/q} \Big| L_p \right\| \tag{61}$$

(*modification if $q=\infty$*) *is an equivalent quasi-norm in $F^s_{p,q}$.*

Proof. We have to prove that the quasi-norm in (61) can be estimated from above by the quasi-norm in (58). For this purpose we estimate $\sup_{0<|h|\leqslant t}$ from above by $\sum_{j=0}^\infty \sup_{2^{-j-1}t\leqslant|h|\leqslant 2^{-j}t}$. Then we have

$$\begin{aligned} &\int_0^1 t^{-sq} \sup_{0<|h|\leqslant t} |(F^{-1}\varphi(h\cdot)\, Ff)(\cdot)|^q \frac{dt}{t} \\ &\quad\leqslant \sum_{j=0}^\infty \int_0^1 t^{-sq} \sup_{2^{-j-1}t\leqslant|h|\leqslant 2^{-j}t} |(F^{-1}\varphi(h\cdot)\, Ff)(\cdot)|^q \frac{dt}{t} \\ &\quad\leqslant \left(\sum_{j=0}^\infty 2^{-jsq}\right) \int_0^1 t^{-sq} \sup_{t/2\leqslant|h|\leqslant t} |(F^{-1}\varphi(h\cdot)\, Ff)(\cdot)|^q \frac{dt}{t}. \end{aligned}$$

This is just the desired estimate.

COROLLARY 5. *Let $0<p<\infty$, $0<q\leqslant\infty$, and $-\infty<s<\infty$. Let s_0 and s_1 be two real numbers with (13) and $s_1>\tilde{\sigma}_p$. Let $\varphi_0(x)$ be the same function as in Theorem 1, including (14) and (18), where $a>n/\min(p,q)$. Let $\varphi^{(1)}(x), \ldots, \varphi^{(N)}(x)$ be N infinitely differentiable complex-valued functions on $R_n-\{0\}$ which satisfy*

$$\sum_{k=1}^N |\varphi^{(k)}(x)|>0 \qquad \text{if } \tfrac{1}{2}\leqslant|x|\leqslant 2 \tag{62}$$

(*Tauberian condition*) *and* (16), (17) *with* $\varphi^{(k)}$ *instead of* φ. *Then*

$$\|F^{-1}\varphi_0 Ff \mid L_p\| + \sum_{k=1}^{N} \left\| \left(\int_0^1 t^{-sq} \, |(F^{-1}\varphi^{(k)}(t\cdot)\, Ff)(\cdot)|^q \frac{dt}{t} \right)^{1/q} \,\middle|\, L_p \right\| \tag{63}$$

and

$$\|F^{-1}\varphi_0 Ff \mid L_p\| + \sum_{k=1}^{N} \left\| \left(\int_0^1 t^{-sq} \sup_{t/2 \leqslant \tau \leqslant t} |(F^{-1}\varphi^{(k)}(\tau\cdot)\, Ff)(\cdot)|^q \frac{dt}{t} \right)^{1/q} \,\middle|\, L_p \right\| \tag{64}$$

are equivalent quasi-norms in $F^s_{p,q}$. *If, in addition*, $s>0$ *then*

$$\|F^{-1}\varphi_0 Ff \mid L_p\| + \sum_{k=1}^{N} \left\| \left(\int_0^1 t^{-sq} \sup_{0 < \tau \leqslant t} |(F^{-1}\varphi^{(k)}(\tau\cdot)\, Ff)(\cdot)|^q \frac{dt}{t} \right)^{1/q} \,\middle|\, L_p \right\| \tag{65}$$

is also an equivalent quasi-norm in $F^s_{p,q}$.

Proof. This corollary is simply the discrete version of Theorem 2 and Corollary 4. The proof is the same.

Remark 12. In particular (63), (64) and, if in addition $s>0$, (65), with $N=1$ and $\varphi^{(1)}=\varphi$ are equivalent quasi-norms in $F^s_{p,q}$, where φ has the meaning of Theorem 1.

COROLLARY 6. *Let* $0<p<\infty$, $0<q\leqslant\infty$, *and* $-\infty<s<\infty$. *Let* $a>n/\min(p,q)$. *Let* s_0 *and* s_1 *be two real numbers with* (13). *Let* $\varphi(t)$ *be an infinitely differentiable complex-valued function on* $R_1-\{0\}$ *which satisfies* (54)–(56). *Then*

$$\left\| \left(\int_{R_n} |h|^{-sq} \, |(F^{-1}\varphi(h\cdot)\, Ff)(\cdot)|^q \frac{dh}{|h|^n} \right)^{1/q} \,\middle|\, L_p \right\| \tag{66}$$

and

$$\left\| \left(\int_0^\infty t^{-sq} \sup_{t/2 \leqslant |h| \leqslant t} |(F^{-1}\varphi(h\cdot)\, Ff)(\cdot)|^q \frac{dt}{t} \right)^{1/q} \,\middle|\, L_p \right\| \tag{67}$$

(modification if $q=\infty$) are equivalent quasi-norms in $\dot{F}^s_{p,q}$. If, in addition, $s>0$ then

$$\left\| \left(\int_0^\infty t^{-sq} \sup_{0<|h|\leqslant t} |(F^{-1}\varphi(h\cdot)\, Ff)(\cdot)|^q \frac{dt}{t} \right)^{1/q} \Big| L_p \right\| \tag{68}$$

(modification if $q=\infty$) is also an equivalent quasi-norm in $\dot{F}^s_{p,q}$.

Remark 13. This corollary is the counterpart of Theorem 2 and Corollary 4 for the homogeneous spaces $\dot{F}^s_{p,q}$. Furthermore there is also an obvious counterpart of Corollary 5.

Further modifications. It is quite clear that some of the above considerations can be generalized and modified. For instance, the C^∞-smoothness assumptions for the functions φ_0 and φ from the two theorems and the subsequent corollaries can be weakened, e.g., in the sense of the Bessel-potential spaces from Corollary 3. Furthermore, one can replace the special family $\{\varphi(hx)\}_{h\in R_n}$ in Theorem 2 by more general families $\{\varphi_h(x)\}_{h\in R_n}$ which satisfy conditions of type (55), (56). However, we wish to describe another modification in some detail because there is a connection with the so-called tent spaces which attracted some attention recently, (cf. [5, 6]), and also with characterizations of the spaces $F^s_{p,q}$ via Lusin functions given by Päivärinta [12, 13, 14] (cf. also [22, 2.12.1]). Let C_1 be the truncated cone

$$C_1 = \{(y,t) \mid y\in R_n, 0<t<1, |y|<t\} \tag{69}$$

in R_{n+1}. Then one has the following modification of Corollary 1, where we now prefer the continuous version: Let $0<p<\infty$, $0<q\leqslant\infty$, and $-\infty<s<\infty$. Let $a>n/\min(p,q)$. Let s_0 and s_1 be two real numbers with (48) and $s_1>\tilde{\sigma}_p$. Let $\varphi_0(x)$ and $\varphi(x)$ be the two functions from Theorem 1 with (14)–(18). Then

$$\|F^{-1}\varphi_0 Ff \mid L_p\| + \left\| \left(\int_{C_1} t^{-sq} |(F^{-1}\varphi(t\cdot)Ff)(x+y)|^q \, dy \frac{dt}{t^{n+1}} \right)^{1/q} \Big| L_p \right\| \tag{70}$$

(modification if $q=\infty$) is an equivalent quasi-norm in $F^s_{p,q}$. The integral over C_1 is taken with respect to (y,t) and the L_p-integral with respect to $x\in R_n$. We outline a proof. For fixed $x\in R_n$ and, say, $t=2^{-j}$ with $j=1,2,\ldots$ we have

$$|(F^{-1}\varphi(t\cdot)\, Ff)(x+y)| \leqslant c(\varphi_j^* f)_a(x), \qquad |y|\leqslant t, \tag{71}$$

where $(\varphi_j^* f)_a(x)$ has the same meaning as in Corollary 1. Hence by this corollary the quasi-norm in (70) can be estimated from above by a quasi-norm in $F^s_{p,q}$. (As in the above considerations there is no problem to replace $\sum_{j=1}^{\infty}\cdots$ in (49) by $\int_0^1 \cdots dt/t$ in the previous sense). In order to prove the reverse estimate we use (38) with $y+z$ instead of y (integration over y), where $|z| \leqslant t = 2^{-j}$. We have (39) with $y+z$ on the left-hand side and y on the right-hand side. We put this estimate in the just-described modified version of (38) and take afterwards the $(\int_{|z|<t} |\cdots|^{q/r}\, dz)^{r/q}$-norm on both sides. Now the rest is the same as at the end of Step 2 of the proof of Theorem 2, and we are through. We add few remarks. (70) is an extension of Päivärinta's characterization of the spaces $F^s_{p,q}$ via Lusin functions; cf. [12, 13, 14] or [22, 2.12.1]. For that purpose one has simply to observe that the integral over C_1 in (70) can be written as

$$\int_0^1 t^{-sq} \frac{1}{|B(x,t)|} \int_{B(x,t)} |(F^{-1}\varphi(t\cdot)\, Ff)(y)|^q\, dy \frac{dt}{t}, \tag{72}$$

where $B(x, t)$ stands for a ball of radius t in R_n centered at $x \in R_n$ with $|B(x, t)|$ as its volume. It is almost obvious that one has a counterpart for the spaces $\dot{F}^s_{p,q}$. One must replace the truncated cone C_1 from (69) by the full cone

$$C_\infty = \{(y, t) \mid y \in R_n, 0 < t < \infty, |y| < t\}. \tag{73}$$

Let $0<p<\infty$, $0<q\leqslant\infty$, and $-\infty<s<\infty$. Let $a>n/\min(p, q)$. Let s_0 and s_1 be two real numbers with (48). Let $\varphi(x)$ be the function from Theorem 1 with (15)–(17). Then

$$\left\| \left(\int_{C_\infty} t^{-sq} |(F^{-1}\varphi(t\cdot)\, Ff)(x+y)|^q\, dy \frac{dt}{t^{n+1}} \right)^{1/q} \,\Big|\, L_p \right\| \tag{74}$$

(modification if $q=\infty$) is an equivalent quasi-norm in $\dot{F}^s_{p,q}$.

2.3. *Spaces of Type* $B^s_{p,q}$

This subsection deals with the spaces $B^s_{p,q}$ and $\dot{B}^s_{p,q}$ from Definition 1 and Remark 4, where again the homogeneous spaces $\dot{B}^s_{p,q}$ are treated as a by-product. We formulate the counterparts of the two preceding subsections and indicate the necessary modifications in the proofs. We use the numbers from (11) and (12), as well as the functions $h(x)$ and $H(x)$ defined at the beginning of Subsection 2.1.

THEOREM 3. *Let* $0<p\leqslant\infty$, $0<q\leqslant\infty$, *and* $-\infty<s<\infty$. *Let* s_0 *and* s_1 *be two real numbers with*

$$s_0 + \tilde{\sigma}_p < s < s_1 \tag{75}$$

and $s_1 > \tilde{\sigma}_p$. *Let* $\varphi_0(x)$ *and* $\varphi(x)$ *be infinitely differentiable complex-valued functions on* R_n *and* $R_n - \{0\}$, *respectively, which satisfy the Tauberian conditions*

$$|\varphi_0(x)| > 0 \qquad \text{if} \quad |x| \leqslant 2 \tag{76}$$

and

$$|\varphi(x)| > 0 \qquad \text{if} \quad \tfrac{1}{2} \leqslant |x| \leqslant 2. \tag{77}$$

Let $\tilde{p} = \min(1, p)$ *and let*

$$\int_{R_n} \left| \left(F^{-1} \frac{\varphi(z)\, h(z)}{|z|^{s_1}} \right)(y) \right|^{\tilde{p}} dy < \infty, \tag{78}$$

$$\sup_{m=1,2,\ldots} 2^{-ms_0\tilde{p}} \int_{R_n} |(F^{-1}\varphi(2^m\cdot)\, H(\cdot))(y)|^{\tilde{p}}\, dy < \infty \tag{79}$$

and

$$\sup_{m=1,2,\ldots} 2^{-ms_0\tilde{p}} \int_{R_n} |(F^{-1}\varphi_0(2^m\cdot)\, H(\cdot))(y)|^{\tilde{p}}\, dy < \infty. \tag{80}$$

Let $\varphi_j(x) = \varphi(2^{-j}x)$ *if* $x \in R_n - \{0\}$ *and* $j = 1, 2, 3, \ldots$. *Then*

$$\left(\sum_{j=0}^{\infty} 2^{jsq}\, \|F^{-1}\varphi_j Ff \mid L_p\|^q \right)^{1/q} \tag{81}$$

(*modification if* $q = \infty$) *and*

$$\|F^{-1}\varphi_0 Ff \mid L_p\| + \left(\int_0^1 t^{-sq}\, \|F^{-1}\varphi(t\cdot)\, Ff \mid L_p\|^q \frac{dt}{t} \right)^{1/q} \tag{82}$$

(*modification if* $q = \infty$) *are equivalent quasi-norms in* $B^s_{p,q}$.

Proof. We modify the proof of Theorem 1. We have again the splitting (21), the estimate (23) and the expression (24). Let $1 \leqslant p \leqslant \infty$. Then we apply the L_p-norm to (23), (24), use (78) with $\tilde{p} = 1$, and apply afterwards the l_q-quasi-norm. Then we obtain the following counterpart of (27),

$$\left(\sum_{j=1}^{\infty} 2^{jsq} \left\| \sum_{l=-\infty}^{K} F^{-1}\varphi_j \rho_{l+j} Ff \mid L_p \right\|^q \right)^{1/q}$$
$$\leqslant c \left(\sum_{m=0}^{\infty} 2^{msq}\, \|F^{-1}\tilde{\rho}_m Ff \mid L_p\|^q \right)^{1/q}. \tag{83}$$

Let $0<p<1$. We use an inequality of Nikol'skij type, cf. [22, (1.3.2/5)] and estimate the integral in (24) from above by

$$c2^{(j+K)n(1/p-1)}\left(\int_{R_n}\left|\left(F^{-1}\frac{\varphi(2^{-j}z)\,h(c2^{-j}z)}{|2^{-j}z|^{s_1}}\right)(y)\right|^p\right.$$
$$\left.\times\;|(F^{-1}2^{s(j+l)}\tilde{\rho}_{j+l}Ff)(x-y)|^p\,dy\right)^{1/p} \tag{84}$$

where c is independent of j. We put this estimate in (23), apply the L_p-quasi-norm, use (78) with $\tilde{p}=p$, apply the l_q-quasi-norm and obtain again (83). Recall that $(\sum b_k)^p \leqslant \sum b_k^p$ for non-negative b_k's. As in the first step of the proof of Theorem 1 we arrive at the following counterpart of (31),

$$\left(\sum_{j=0}^{\infty}2^{jsq}\left\|\sum_{l=-\infty}^{K}F^{-1}\varphi_j\rho_{l+j}Ff\mid L_p\right\|^q\right)^{1/q}\leqslant c\,\|f\mid F^s_{p,q}\|. \tag{85}$$

In precisely the same way the second step of the proof of Theorem 1 can be modified. We have to use (79), (80), and obtain

$$\left(\sum_{j=0}^{\infty}2^{jsq}\left\|\sum_{l=K+1}^{\infty}F^{-1}\varphi_j\rho_{l+j}Ff\mid L_p\right\|^q\right)^{1/q}\leqslant c2^{-K(s-s_0)}\,\|f\mid F^s_{p,q}\| \tag{86}$$

as the counterpart of (36). The constant c is independent of K. However (85), (86) prove that the quasi-norm in (81) can be estimated from above by $c\,\|f\mid F^s_{p,q}\|$. In order to prove the reverse inequality we modify Step 3 of the proof of Theorem 1. We have (42) where now $0<r<\min(1,p)$ is sufficient. We use the usual (sacalar) Hardy–Littlewood maximal inequality with respect to the $L_{p/r}$-norm and apply afterwards the $l_{q/r}$-norm. By the same arguments as in Step 3 of the proof of Theorem 1 we obtain that $\|f\mid F^s_{p,q}\|$ can be estimated from above by the quasi-norm in (81). The necessary modifications in order to ıncorporate the quasi-norm in (82) have been described in the proof of Theorem 2.

Remark 14. In contrast to the proof of Theorem 1 we avoided the technique of maximal functions completely. The advantage of (78)–(80) compared with (16)–(18) will be clear later on when we discuss equivalent quasi-norms where differences Δ_h^M are involved.

Next we formulate the counterparts of the Corollaries 1–3. In particular, the maximal function $(\varphi_j^* f)_a$ has the same meaning as in Corollɛ y 1.

COROLLARY 7. *Let* $0<p\leqslant\infty$, $0<q\leqslant\infty$, *and* $-\infty<s<\infty$. *Let* $a>n/p$. *Let* s_0 *and* s_1 *be two real numbers with* $s_0+a<s<s_1$ *and* $s_1>\tilde{\sigma}_p$.

Let $\varphi_0(x)$ and $\varphi(x)$ be two infinitely differentiable functions on R_n and $R_n - \{0\}$, respectively, which satisfy (76), (77), *and also* (16)–(18) *with the above number $a > n/p$. Then*

$$\left(\sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_j^* f)_a \mid L_p\|^q \right)^{1/q} \tag{87}$$

(*modification if $q = \infty$*) *is an equivalent quasi-norm in $B^s_{p,q}$.*

Proof. One can use the proof of Theorem 1 with the modifications described in the proof of Corollary 1.

Remark 15. This is the counterpart of Corollary 1. The conditions (78)–(80) are less restrictive than (16)–(18) with $a > n/p$, respectively. This is clear if $p \geqslant 1$ and it follows from Hölder's inequality if $p < 1$.

Again we take the opportunity to formulate the corresponding results for the homogeneous spaces $\dot{B}^s_{p,q}$.

COROLLARY 8. *Let $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, and $-\infty < s < \infty$. Let $a > n/p$. Let $\varphi(x)$ be an infinitely differentiable complex-valued function on $R_n - \{0\}$ which satisfies* (15), *and let $\varphi_j(x)$ be given by* (19) *where j is an arbitrary integer.*

(i) *Let s_0 and s_1 be two real numbers with* (75). *Let* (78) *and* (79) *be satisfied. Then*

$$\left(\sum_{j=-\infty}^{\infty} 2^{jsq} \|F^{-1}\varphi_j Ff \mid L_p\|^q \right)^{1/q} \tag{88}$$

(*modification if $q = \infty$*) *is an equivalent quasi-norm in $\dot{B}^s_{p,q}$.*

(ii) *Let s_0 and s_1 be two real numbers with* (48) *and let* (16) *and* (17) *be satisfied, where $a > n/p$ has the above meaning. Then*

$$\left(\sum_{j=-\infty}^{\infty} 2^{jsq} \|(\varphi_j^* f)_a \mid L_p\|^q \right)^{1/q} \tag{89}$$

(*modification if $q = \infty$*) *is an equivalent quasi-norm in $\dot{B}^s_{p,q}$.*

Remark 16. This is the counterpart of Corollary 2 and it extends Theorem 3 and Corollary 7 from the non-homogeneous spaces to the homogeneous ones.

The conditions for $\varphi_0(x)$ and $\varphi(x)$ from Corollary 7 can be reformulated in the sense of Corollary 3. We give the corresponding reformulation for the conditions (78)–(80). Let again H^σ_2 be the usual Bessel-potential spaces on R_n (Sobolev spaces if σ is a natural number). σ_p has been defined in (11).

COROLLARY 9. *Let the hypotheses of Theorem* 3 *for the numbers* p, q, s, s_0, *and* s_1 *be fulfilled. Furthermore let the conditions* (16′)–(18′) *from Corollary* 3 *with* $\sigma > \sigma_p$ *be satisfied. Let* $\varphi_j(x)$ *be the same functions as in Theorem* 3. *Then* (81) *and* (82) *are equivalent quasi-norms in* $B^s_{p,q}$.

Proof. The corollary follows from Theorem 3 and (53).

Our next task is to carry over the results obtained in Subsections 2.2 for the spaces $F^s_{p,q}$ to the spaces $B^s_{p,q}$.

THEOREM 4. *Let* $0<p\leqslant\infty$, $0<q\leqslant\infty$, *and* $-\infty<s<\infty$. *Let* s_0 *and* s_1 *be two real numbers with* (75) *and* $s_1>\tilde{\sigma}_p$. *Let* $\varphi_0(x)$ *be the same function as in Theorem* 3, *including* (76) *and* (80), *where again* $\tilde{p}=\min(1,p)$. *Let* $\varphi(t)$ *be an infinitely differentiable complex-valued function on* $R_1-\{0\}$ *which satisfies the Tauberian condition* $|\varphi(t)|>0$ *if* $\frac{1}{8}<t<8$ *and*

$$\sup_{1\leqslant|y|\leqslant 2}\int_{R_n}\left|\left(F^{-1}\frac{\varphi(\gamma z)\,h(z)}{|z|^{s_1}}\right)(y)\right|^{\tilde{p}}dy<\infty, \tag{90}$$

$$\sup_{1\leqslant|y|\leqslant 2}\ \sup_{m=1,2,\dots}\ 2^{-ms_0\tilde{p}}\int_{R_n}|(F^{-1}\varphi(2^m\gamma z)\,H(z))(y)|^{\tilde{p}}\,dy<\infty. \tag{91}$$

Then

$$\|F^{-1}\varphi_0Ff\mid L_p\|+\left(\int_{|h|\leqslant 1}|h|^{-sq}\,\|F^{-1}\varphi(h\cdot)\,Ff\mid L_p\|^q\frac{dh}{|h|^n}\right)^{1/q} \tag{92}$$

and

$$\|F^{-1}\varphi_0Ff\mid L_p\|+\left(\int_0^1 t^{-sq}\sup_{t/2\leqslant|h|\leqslant t}\|F^{-1}\varphi(h\cdot)\,Ff\mid L_p\|^q\frac{dt}{t}\right)^{1/q} \tag{93}$$

(*modification if* $q=\infty$) *are equivalent quasi-norms in* $B^s_{p,q}$. *If, in addition,* $s>0$ *then*

$$\|F^{-1}\varphi_0Ff\mid L_p\|+\left(\int_0^1 t^{-sq}\sup_{0<|h|\leqslant t}\|F^{-1}\varphi(h\cdot)\,Ff\mid L_p\|^q\frac{dt}{t}\right)^{1/q} \tag{94}$$

(*modification if* $q=\infty$) *is an equivalent quasi-norm in* $B^s_{p,q}$.

Proof. One has to modify the proof of Theorem 3 which, in turn, is based on the proof of Theorem 1 in the sense of the proof of Theorem 2. We omit the details. Then we obtain that (92) and (93) are equivalent quasi-norms in $B^s_{p,q}$. If $s>0$ then we can use the arguments from the proof of Corollary 4 in order to show that (94) is also an equivalent quasi-norm.

Remark 17. The above theorem is the counterpart of Theorem 2 and Corollary 4. Furthermore, it is now obvious that one has also a counterpart of Corollary 5. In other words, if one replaces the function $\varphi(t)$ in Theorem 4 by N infinitely differentiable complex-valued functions $\varphi^{(1)}(x), ..., \varphi^{(N)}(x)$ with (62) and the respective counterparts of (78), (79), then

$$\|F^{-1}\varphi_0 Ff \mid L_p\| + \sum_{k=1}^{N} \left(\int_0^1 t^{-sq} \|F^{-1}\varphi^{(k)}(t\cdot)\, Ff \mid L_p\|^q \frac{dt}{t} \right)^{1/q} \tag{95}$$

and

$$\|F^{-1}\varphi_0 Ff \mid L_p\| + \sum_{k=1}^{N} \left(\int_0^1 \sup_{t/2 \leqslant \tau \leqslant t} \|F^{-1}\varphi^{(k)}(\tau\cdot)\, Ff \mid L_p\|^q \frac{dt}{t} \right)^{1/q} \tag{96}$$

(modification if $q = \infty$) are equivalent quasi-norms in $B^s_{p,q}$. If $s > 0$ then one can replace $\sup_{t/2 \leqslant \tau \leqslant t}$ in (96) by $\sup_{0 < \tau \leqslant t}$. Furthermore one can describe a counterpart of (70) for $B^s_{p,q}$.

Finally we formulate the counterpart both of Theorem 4 and Corollary 6 for the homogeneous spaces $\dot{B}^s_{p,q}$.

COROLLARY 10. *Let* $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, *and* $-\infty < s < \infty$. *Let* s_0 *and* s_1 *be two real numbers with* (75). *Let* $\varphi(t)$ *be an infinitely differentiable complex-valued function on* $R_1 - \{0\}$ *which satisfies* (54) *and* (90), (91). *Then*

$$\left(\int_{R_n} |h|^{-sq} \|F^{-1}\varphi(h\cdot)\, Ff \mid L_p\|^q \frac{dh}{|h|^n} \right)^{1/q} \tag{97}$$

and

$$\left(\int_0^\infty t^{-sq} \sup_{t/2 \leqslant |h| \leqslant t} \|F^{-1}\varphi(h\cdot)\, Ff \mid L_p\|^q \frac{dt}{t} \right)^{1/q} \tag{98}$$

(*modifications if* $q = \infty$) *are equivalent quasi-norms in* $\dot{B}^s_{p,q}$. *If, in addition,* $s > 0$, *then*

$$\left(\int_0^\infty t^{-sq} \sup_{0 < |h| \leqslant t} \|F^{-1}\varphi(h\cdot)\, Ff \mid L_p\|^q \frac{dt}{t} \right)^{1/q} \tag{99}$$

(*modification if* $q = \infty$) *is also an equivalent quasi-norm in* $\dot{B}^s_{p,q}$.

Remark 18. Furthermore there are also obvious counterparts of the quasi-norms (95) and (96) for the homogeneous spaces $\dot{B}^s_{p,q}$.

3. Concrete Characterizations

3.1. *Characterizations via Differences and Derivatives*

As in Section 2 we are mostly interested in characterizations for the non-homogeneous spaces $B^s_{p,q}$ and $F^s_{p,q}$. However, we shall formulate the corresponding results for the homogeneous spaces $\dot{B}^s_{p,q}$ and $\dot{F}^s_{p,q}$, too. First, we introduce a few notations. If $\alpha = (\alpha_1, ..., \alpha_n)$ is a multi-index, i.e., the α_j's are non-negative integers, then $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$ with $|\alpha| = \alpha_1 + \cdots \alpha_n$ stands for the derivatives. Let

$$(\Delta_h^1 f)(x) = f(x+h) - f(x) \qquad \text{and} \qquad \Delta_h^M = \Delta_h^1 \Delta_h^{M-1} \tag{100}$$

with $M = 2, 3, ...,$ be the usual differences, where $x \in R_n$ and $h \in R_n$. If $h = (t, 0, ..., 0)$ with $-\infty < t < \infty$ then we write $\Delta_h^M = \Delta_{t,1}^M$ for the partial differences with respect to the first direction of the co-ordinates. Similarly $\Delta^M_{t,k}$ with $k = 2, ..., n$. Recall that the numbers σ_p, etc., have been defined in (11), (12).

THEOREM 5. *Let $0 < p \leqslant \infty$, $0 < q \leqslant \infty$ and $\tilde{\sigma}_p < s < M$, where M is a natural number. Then*

$$\|f \mid L_p\| + \left(\int_{|h| \leqslant 1} |h|^{-sq} \, \|\Delta_h^M f \mid L_p\|^q \frac{dh}{|h|^n} \right)^{1/q}, \tag{101}$$

$$\|f \mid L_p\| + \left(\int_0^1 t^{-sq} \sup_{0 < |h| \leqslant t} \|\Delta_h^M f \mid L_p\|^q \frac{dt}{t} \right)^{1/q}, \tag{102}$$

$$\|f \mid L_p\| + \sum_{k=1}^n \left(\int_0^1 t^{-sq} \, \|\Delta_{t,k}^M f \mid L_p\|^q \frac{dt}{t} \right)^{1/q}, \tag{103}$$

and

$$\|f \mid L_p\| + \sum_{k=1}^n \left(\int_0^1 t^{-sq} \sup_{0 < \tau \leqslant t} \|\Delta_{\tau,k}^M f \mid L_p\|^q \frac{dt}{t} \right)^{1/q} \tag{104}$$

are equivalent quasi-norms in $B^s_{p,q}$.

Proof. We use Theorem 4 with $\varphi_0(x) = 1$ and $\varphi(t) = (e^{ivt} - 1)^M$, where v is a positive number. Then we have (76) and $|\varphi(t)| > 0$ for $\frac{1}{8} < t < 8$ if $v > 0$ is small. Recall that $F^{-1}\,(e^{ihx} - 1)^M = \Delta_h^M$. Then (75), (80), and (91) with $s_0 = 0$ are satisfied. Finally we fix s_1 with $s_1 > s$. If M is large then (90) is fulfilled; cf. Corollary 9 or (53). Then (92) and (94) prove that (101) and (102) with large natural numbers M are equivalent quasi-norms in $B^s_{p,q}$ where the integration over $|h| \leqslant 1$ and $0 < t < 1$ can be replaced by an integration over $|h| \leqslant \mu$ and $0 < t < \mu$ respectively, where $\mu > 0$ is an arbitrary number (cf. also Subsect. 3.2, where we return to just this point in Corollary 14). This result can be extended to any natural number M with $M > s_1$; cf. Remark 19, where we discuss this point. In order to prove that (103) and (104) are equivalent quasi-norms in $B^s_{p,q}$ we choose $\varphi^{(k)}(x) = (e^{ivx_k} - 1)^M$ with $v > 0$, $x = (x_1, ..., x_n)$ and $k = 1, ..., n$ instead of the above function φ. Then (62) is satisfied if $v > 0$ is small. Then Remark 17 proves that the expressions in (103) and (104) are equivalent quasi-norms in $B^s_{p,q}$ provided that M is large. The fact that even $M > s_1$ is sufficient follows from Remark 19 below.

Remark 19. The above proof covers only the case if M in (101)–(104) is large. We have to show that the expressions in (101)–(104) are equivalent quasi-norms in $B^s_{p,q}$ for every natural number M with $M > s$. There are two possibilities where we restrict ourselves to the quasi-norm in (101). The other cases can be treated similarly. (i) The integration over $|h| \leqslant 1$ in (101) can be replaced by an integration over R_n and let $\|f \mid B^s_{p,q}\|_M$ be the quasi-norm from (101) modified in this way. Let $s < M$. Then $\|f \mid B^s_{p,q}\|_{M+1} \leqslant c\, \|f \mid B^s_{p,q}\|_M$ is obvious. Recall

$$(\Delta^M_h f)(x) = 2^{-M}(\Delta^M_{2h} f)(x) + \Delta^{M+1}_h \left(\sum a_l f(x + lh)\right), \tag{105}$$

where $\sum$ stands for a finite sum and a_l are real numbers; cf. [22, (45) in 2.5.9]. If one puts (105) in $\|f \mid B^s_{p,q}\|_M$ then we have

$$\|f \mid B^s_{p,q}\|_M \leqslant c\, \|f \mid B^s_{p,q}\|_{M+1} + 2^{s-M}\, \|f \mid B^s_{p,q}\|_M. \tag{106}$$

Because $M > s$, this proves that $\|f \mid B^s_{p,q}\|_M$ and $\|f \mid B^s_{p,q}\|_{M+1}$ are equivalent. The rest is a matter of mathematical induction. (ii) A second possibility is to modify (55) and (90), where we replace $|z|^{s_1}$ by $(\gamma z)^M$ where M is the above natural number. This has the consequence that one must replace $|x|^{s_1}$ in (22) by $(\gamma x)^M$. Because $(\gamma x)^M$ is a polynomial one has no problems to modify the proof of Theorem 2 and, hence, of Theorem 4. However $(\gamma z)^{-M}(e^{iv\gamma x} - 1)^M$ is an analytic function and both (55) and (90) are satisfied for any $s_1 = M > s$. These arguments work also for the spaces $F^s_{p,q}$ which will be used later on.

COROLLARY 11. *Let $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, and $\tilde{\sigma}_p < s - L < M$ where L and M are natural numbers. Then*

$$\|f \mid L_p\| + \sum_{|\alpha| = L} \left(\int_{|h| \leqslant 1} |h|^{-(s-L)q}\, \|\Delta^M_h D^\alpha f \mid L_p\|^q \frac{dh}{|h|^n}\right)^{1/q} \tag{107}$$

and

$$\|f \mid L_p\| + \sum_{k=1}^{n} \left(\int_0^1 t^{-(s-L)q} \left\| \Delta_{t,k}^M \frac{\partial^L f}{\partial x_k^L} \,\middle|\, L_p \right\|^q \frac{dt}{t} \right)^{1/q} \tag{108}$$

are equivalent quasi-norms in $B^s_{p,q}$.

Proof. Let $\varphi^{(k)}(x) = x_k^L (e^{ivx_k} - 1)^M$ with $k = 1, ..., n$ and $v > 0$. Let again $\varphi_0(x) = 1$. If $v > 0$ is small then (62) is satisfied. We use (75) with $s_0 = L$ and $s_1 = L + M$. Then the respective counterparts of (78), (79) are fulfilled, where in the latter case one can apply the arguments from Remark 19. Now (95) proves that (108) is an equivalent quasi-norm where we again refer to Subsection 3.2 as far as the replacement of $\int_{|h| \leqslant v}$ by $\int_{|h| \leqslant 1}$ is concerned. Now we complement the functions $\varphi^{(k)}(x)$ by all functions $x_1^{L_1} \cdots x_n^{L_n} (e^{ivx_k} - 1)^M$ with $L_1 + \cdots + L_n = L$ and apply again (95). This shows that (108) with $\sum_{|\alpha| = L} \|\Delta_{t,k}^M D^\alpha f \mid L_p\|$ instead of $\sum_{k=1}^n \|\Delta_{t,k}^M (\partial^L f / \partial x_k^L) \mid L_p\|$ is also an equivalent quasi-norm. Because (101) and (103) are equivalent quasi-norms it follows from the last observation that (107) is also an equivalent quasi-norm.

Remark 20. Of course the corresponding counterparts of (102) and (104) in the sense of (107) and (108), respectively, are also equivalent quasi-norms in $B^s_{p,q}$.

THEOREM 6. *Let* $0 < p < \infty$, $0 < q \leqslant \infty$ *and* $n/\min(p, q) < s < M$, *where* M *is a natural number. Then*

$$\|f \mid L_p\| + \left\| \left(\int_{|h| \leqslant 1} |h|^{-sq} \, |(\Delta_h^M f)(\cdot)|^q \frac{dh}{|h|^n} \right)^{1/q} \,\middle|\, L_p \right\|, \tag{109}$$

$$\|f \mid L_p\| + \left\| \left(\int_0^1 t^{-sq} \sup_{0 < |h| \leqslant t} |(\Delta_h^M f)(\cdot)|^q \frac{dt}{t} \right)^{1/q} \,\middle|\, L_p \right\|, \tag{110}$$

$$\|f \mid L_p\| + \sum_{k=1}^{n} \left\| \left(\int_0^1 t^{-sq} \, |(\Delta_{t,k}^M f)(\cdot)|^q \frac{dt}{t} \right)^{1/q} \,\middle|\, L_p \right\|, \tag{111}$$

and

$$\|f \mid L_p\| + \sum_{k=1}^{n} \left\| \left(\int_0^1 t^{-sq} \sup_{0 < \tau \leqslant t} |(\Delta_{\tau,k}^M f)(\cdot)|^q \frac{dt}{t} \right)^{1/q} \,\middle|\, L_p \right\| \tag{112}$$

are equivalent quasi-norms in $F^s_{p,q}$.

Proof. We use the same functions φ_0, φ, and $\varphi^{(k)}$ as in the proof of Theorem 5. Let, in addition $\tilde{\sigma}_{p,q} + (n/\min(p, q)) < s$. Then we choose $s_0 = a > n/\min(p, q)$ with $s_0 + \tilde{\sigma}_{p,q} < s$ and $s_1 = M$. We have (13), (14), and

(18). Furthermore (56) holds where we need now $s_0 = a$. As far as (55) is concerned we use the arguments from Remark 19. Hence we can apply Theorem 2 and Corollary 4 which cover (109) and (110). As far as the replacement of $\int_{|h| \leqslant v}$ and $\int_0^v$ by $\int_{|h| \leqslant 1}$ and $\int_0^1$, respectively, are concerned we refer again to Subsection 3.2, Corollary 14. The remaining quasi-norms in (111) and (112) follow from Corollary 5 in the same way as above. Hence the theorem is proved if $\tilde{\sigma}_{p,q} + (n/\min(p,q)) < s$. If we have only $s > n/\min(p,q)$ and $a = s_0 < s$, then it follows from Remark 11 that the quasi-norms in (109)–(112) can be estimated from above by the $F^s_{p,q}$-quasi-norm. We postpose the coverse inequality to Remark 23.

COROLLARY 12. *Let* $0 < p < \infty$, $0 < q \leqslant \infty$, *and* $n/\min(p,q) < s - L < M$, *where L and M are natural numbers. Then*

$$\|f \mid L_p\| + \sum_{|\alpha| = L} \left\| \left(\int_{|h| \leqq L} |h|^{-(s-L)q} \, |(\Delta_h^M D^\alpha f)(\cdot)|^q \frac{dh}{|h|^n} \right)^{1/q} \,\middle|\, L_p \right\| \tag{113}$$

and

$$\|f \mid L_p\| + \sum_{k=1}^{n} \left\| \left(\int_0^1 t^{-(s-L)q} \left| \left(\Delta_{t,k}^M \frac{\partial^L f}{\partial x_k^L} \right)(\cdot) \right|^q \frac{dt}{t} \right)^{1/q} \,\middle|\, L_p \right\| \tag{114}$$

are equivalent quasi-norms in $F^s_{p,q}$.

Proof. The proof follows from the preceding theorem and the arguments of the proof of Corollary 11.

Remark 21. The conditions for s and M in Theorem 5 and Corollary 11 and at least for $q \geqslant p$ also in Theorem 6 are natural. Characterizations of type (110) cannot be expected if $s < n/p$, because in that case $F^s_{p,q}$ contains essentially unbounded functions for which the corresponding expression is infinite.

The above two theorems and the subsequent corollaries have more or less obvious counterparts for the homogeneous spaces $\dot{B}^s_{p,q}$ and $\dot{F}^s_{p,q}$. One has to use Corollaries 6 and 10, and also the modifications indicated in Remark 19. We restrict ourselves to an example.

COROLLARY 13. (i) *Let* $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, *and* $\tilde{\sigma}_p < s - L < M$ *where L is a non-negative integer and M is a natural number. Then*

$$\sum_{|\alpha| = L} \left(\int_{R_n} |h|^{-(s-L)q} \, \|\Delta_h^M D^\alpha f \mid L_p\|^q \frac{dh}{|h|^n} \right)^{1/q} \tag{115}$$

is an equivalent quasi-norm in $\dot{B}^s_{p,q}$.

(ii) *Let* $0<p<\infty$, $0<q\leqslant\infty$, *and* $n/\min(p,q)<s-L<M$, *where* L *is a non-negative integer and* M *is a natural number. Then*

$$\sum_{|\alpha|=L}\left\|\left(\int_{R_n}|h|^{-(s-L)q}\,|(\Delta_h^M I^\alpha f)(\cdot)|^q\frac{dh}{|h|^n}\right)^{1/q}\,\Big|\,L_p\right\| \tag{116}$$

is an equivalent quasi-norm in $\dot{F}^s_{p,q}$.

3.2. *Characterizations via Differences and Derivatives: Complements*

In the preceding subsection we used functions of the type $\varphi(t)=(e^{ivt}-1)^M$ in order to reduce the assertions from Subsection 3.1 to the respective theorems and corollaries in Section 2. It was convenient to assume that $v>0$ is small. The direct application of the results of Section 2 yields, for example, the quasi-norms in (109) and (110) with $\int_{|h|\leqslant v}$ and $\int_0^v$ instead of $\int_{|h|\leqslant 1}$ and $\int_0^1$, respectively. However, by simultaneous dilations $h\to ch$ and $x\to c'x$ with $c>0$ and $c'>0$ it follows that for any number $\mu>0$ the expression in (109) with $\int_{|h|\leqslant\mu}$ instead of $\int_{|h|\leqslant 1}$ is an equivalent quasi-norm in $F^s_{p,q}$. Of course, a similar remark holds true for all the other quasi-norms in Subsection 3.1 for the non-homogeneous spaces $B^s_{p,q}$ and $F^s_{p,q}$. One can ask whether $\int_{|h|\leqslant\mu}\cdots dh/|h|^n$ can be replaced by $\int_{R_n}\cdots dh/|h|^n$, etc. An affirmative answer can be obtained by direct calculations. But we prefer a more general setting which is essentially a comparison of homogeneous and non-homogeneous spaces and which can also be applied to other situations than those ones dealt with in the preceding subsection. Let $\varphi(x)\in S$ with (2) and, say, $\sum_{j=-\infty}^{\infty}\varphi_j(x)=1$ if $x\neq 0$, where $\varphi_j(x)=\varphi(2^{-j}x)$. Let $\varphi^0(x)\in S$ with $\operatorname{supp}\varphi^0\subset\{y\mid |y|\leqslant 2\}$ and $\varphi^0(x)=1$ if $|x|\leqslant 1$. Recall that $\tilde{\sigma}_p$ has been defined in (12).

PROPOSITION. (i) *Let* $0<p\leqslant\infty$, $0<q\leqslant\infty$, *and* $s>\tilde{\sigma}_p$. *Then*

$$\|F^{-1}\varphi^0 Ff\mid L_p\|+\left(\sum_{j=-\infty}^{\infty}2^{sjq}\,\|F^{-1}\varphi_j Ff\mid L_p\|^q\right)^{1/q} \tag{117}$$

and

$$\|f\mid L_p\|+\left(\sum_{j=-\infty}^{\infty}2^{sjq}\,\|F^{-1}\varphi_j Ff\mid L_p\|^q\right)^{1/q} \tag{118}$$

(*modification if* $q=\infty$) *are equivalent quasi-norms in* $B^s_{p,q}$.

(ii) *Let* $0<p<\infty$, $0<q\leqslant\infty$, *and* $s>\tilde{\sigma}_p$. *Then*

$$\|F^{-1}\varphi^0 Ff\mid L_p\|+\left\|\left(\sum_{j=-\infty}^{\infty}2^{sjq}\,|(F^{-1}\varphi_j Ff)(\cdot)|^q\right)^{1/q}\,\Big|\,L_p\right\| \tag{119}$$

and

$$\|f \mid L_p\| + \left\| \left(\sum_{j=-\infty}^{\infty} 2^{sjq} \, |(F^{-1}\varphi_j Ff)(\cdot)|^q \right)^{1/q} \Big| \, L_p \right\| \tag{120}$$

(*modification if* $q=\infty$) *are equivalent quasi-norms in* $F^s_{p,q}$.

Proof. Step 1. In order to prove that (117) and (119) are equivalent quasi-norms in $B^s_{p,q}$ and $F^s_{p,q}$, respectively, it is sufficient to show that there exists a constant $c>0$ such that

$$\|F^{-1}\varphi_j Ff \mid L_p\| \leqslant c 2^{-j\tilde{\sigma}_p} \|F^{-1}\varphi_0 Ff \mid L_p\| \tag{121}$$

holds for all $j=-1, -2, -3, \ldots$, cf. (6) and (7). However, we have

$$\begin{aligned} \|F^{-1}\varphi_j Ff \mid L_p\| &= \|F^{-1}\varphi_j FF^{-1}\varphi^0 Ff \mid L_p\| \\ &\leqslant c \, \|F^{-1}\varphi_j \mid L_{\tilde{p}}\| \, \|F^{-1}\varphi^0 Ff \mid L_p\| \end{aligned} \tag{122}$$

with $\tilde{p}=\min(1,p)$, cf. [22, Proposition 1.5.1]. Because $(F^{-1}\varphi_j)(x) = 2^{jn}(F^{-1}\varphi)(2^j x)$ we obtain (121).

Step 2. We prove that (120) is an equivalent quasi-norm in $F^s_{p,q}$. We have

$$\|f \mid L_p\| \leqslant c \, \|F^{-1}\varphi^0 Ff \mid L_p\| + c \left(\sum_{j=1}^{\infty} \|F^{-1}\varphi_j Ff \mid L_p\|^p \right)^{1/p} \tag{123}$$

if $0<p\leqslant 1$ and a corresponding estimate if $1<p<\infty$. Now, (119) and (123) prove that (120) can be estimated from above by $c \, \|f \mid F^s_{p,q}\|$. We prove the reverse inequality. Because $s>\tilde{\sigma}_p$ we have

$$\begin{aligned} \|F^{-1}\varphi^0 Ff \mid L_p\| \leqslant{} & c_\varepsilon \, \|f \mid L_p\| + \varepsilon \, \|F^{-1}\varphi^0 Ff \mid L_p\| \\ & + \varepsilon \left\| \left(\sum_{j=1}^{\infty} 2^{sjq} \, |(F^{-1}\varphi_j Ff)(\cdot)|^q \right)^{1/q} \Big| \, L_p \right\|, \end{aligned} \tag{124}$$

where $\varepsilon>0$ is at our disposal; cf. [22, 2.5.9, formula (37)]. We choose $\varepsilon=\frac{1}{2}$. Then it follows that $\|f \mid F^s_{p,q}\|$ can be estimated from above by (120). In the same way one obtains that (118) is an equivalent quasi-norm in $B^s_{p,q}$.

Remark 22. Expressions (118) and (120) can be written as

$$\|f \mid L_p\| + \|f \mid \dot{B}^s_{p,q}\| \tag{125}$$

and

$$\|f \mid L_p\| + \|f \mid \dot{F}^s_{p,q}\|, \tag{126}$$

respectively; cf. Remark 4. This shows how the homogeneous and the non-homogeneous spaces are connected if the hypotheses of the above proposition are satisfied.

COROLLARY 14. (i) *Let* $0<p\leqslant\infty$, $0<q\leqslant\infty$, *and* $\tilde{\sigma}_p<s<M$, *where* M *is a natural number. Let* $0<v<\infty$. *Then* (101) *with* $\int_{|h|\leqslant v}\cdots dh/|h|^n$ *or with* $\int_{R_n}\cdots dh/|h|^n$ *instead of* $\int_{|h|\leqslant 1}\cdots dh/|h|^n$, *and* (102)–(104) *with* $\int_0^v\cdots dt/t$ *or with* $\int_0^\infty\cdots dt/t$ *instead of* $\int_0^1\cdots dt/t$ *are equivalent quasi-norms in* $B^s_{p,q}$.

(ii) *Let* $0<p<\infty$, $0<q\leqslant\infty$ *and* $n/\min(p,q)<s<M$, *where* M *is a natural number. Let* $0<v<\infty$. *Then* (109) *with* $\int_{|h|\leqslant v}\cdots dh/|h|^n$ *or with* $\int_{R_n}\cdots dh/|h|^n$ *instead of* $\int_{|h|\leqslant 1}\cdots dh/|h|^n$, *and* (110)–(112) *with* $\int_0^v\cdots dt/t$ *or with* $\int_0^\infty\cdots dt/t$ *instead of* $\int_0^1\cdots dt/t$ *are equivalent quasi-norms in* $F^s_{p,q}$.

Proof. If $v>0$ is small then the above claims are covered by the proofs of the Theorems 5 and 6. On the other hand in Corollary 13 we gave examples of equivalent quasi-norms in $\dot{B}^s_{p,q}$ and $\dot{F}^s_{p,q}$. Using these results the above proposition and Remark 22 it follows that (101) and (109) with $\int_{R_n}\cdots dh/|h|^n$ instead of $\int_{|h|\leqslant 1}\cdots dh/|h|^n$ are equivalent quasi-norms in $B^s_{p,q}$ and $F^s_{p,q}$, respectively. Similar for the remaining quasi-norms. However, as far as the spaces $F^s_{p,q}$ are concerned we must add the same remark as at the end of the proof of Theorem 6: The above considerations work if $\tilde{\sigma}_{p,q}+(n/\min(p,q))<s$. If we know only $s>n/\min(p,q)$ then it follows from the final remarks at the end of the proof of Theorem 6 and their counterparts for the homogeneous spaces that at least all the quasi-norms in Theorem 6 and in part(ii) of the above corollary can be estimated from above by $c\,\|f\mid F^s_{p,q}\|$. We postpone the converse inequality to Remark 23.

3.3. *Characterizations via Weighted Means of Differences and Derivatives, the Localization Principle*

We modify the considerations of Subsection 3.1, but we restrict ourselves to the non-homogeneous spaces $B^s_{p,q}$ and $F^s_{p,q}$. We deal with two versions of weighted means of differences. The first version gives the possibility to replace the assumption $s>n/\min(p,q)$ in Theorem 6 and Corollary 14(ii) for the spaces $F^s_{p,q}$ by the more natural assumption $s>\tilde{\sigma}_{p,q}$, where $\tilde{\sigma}_{p,q}$ has been defined by (12). The price to pay is the replacement of the differences Δ^M_h by weighted means of differences; cf. also Remark 21. One can do the same for the spaces $B^s_{p,q}$ in the sense of Theorem 5. But there is no chance to improve the condition $s>\tilde{\sigma}_p$. So we restrict ourselves to the spaces $F^s_{p,q}$, at least as far as the first version of weighted means is concerned.

Let $g\in S$ be non-negative and rotation-invariant (i.e., $g(x)$ depends only on $|x|$) with $g(0)>0$ and supp Fg compact. Then

$$[K^M(g,t)f](x)=\int_{R_n} g(h)(\Delta^M_{th}f)(x)\,dh,\qquad x\in R_n, \tag{127}$$

are weighted means of differences where M is a natural number and $t>0$. As the subsequent considerations show the assumptions for g are convenient, but they can be weakened.

THEOREM 7. *Let $0<p<\infty$, $0<q\leqslant\infty$, and $\tilde{\sigma}_{p,q}<s<M$, where M is a natural number. Let $0<v\leqslant\infty$. Let g be the above function and let $K^M(g,t)f$ be given by* (127). *Then*

$$\|f \mid L_p\| + \left\| \left(\int_0^v t^{-sq} \, |(K^M(g,t)f)(\cdot)|^q \frac{dt}{t} \right)^{1/q} \Bigm| L_p \right\| \tag{128}$$

is an equivalent quasi-norm in $F^s_{p,q}$.

Proof. Let $\varphi_0(x)=1$ and

$$\varphi(x) = \int_{R_n} g(h)(e^{ivxh}-1)^M \, dh, \qquad x\in R_n, \tag{129}$$

where xh stands for the scalar product of $x\in R_n$ and $h\in R_n$, and $0<v<\infty$. We use Theorem 1 with $s_0=0$ and $s_1=M$ if M is even and $s_1=M+1$ if M is odd. Then (13), (14), (18) and $s_1>\tilde{\sigma}_p$ are satisfied. As for (15) we have

$$\left| \varphi(x) - \int_{|h|\leqslant T} g(h)(e^{ivxh}-1)^M \, dh \right| \leqslant cT^{-M-2}, \tag{130}$$

where c is independent of T and v. On the other hand if $|x|\leqslant 2$ and $v=b/T$ where $b>0$ is small then

$$\int_{|h|\leqslant T} g(h)(e^{ivxh}-1)^M \, dh = cT^{-M} \int_{|h|\leqslant T} g(h)(xh)^M (1+o(1)) \, dh. \tag{131}$$

If M is even then the integral over $g(h)(xh)^M$ in (131) is positive and (130) and (131) yield (15). If M is odd then the integral over $g(h)(xh)^M$ vanishes but not the integral over $g(h)(xh)^{M+1}$. Then one has (131) with T^{-M-1} instead of T^{-M}. Again (15) follows from (130) and this modified equation (131). We have to check (16) and (17). Because $g(h)$ is rotation-invariant, $\varphi(x)$ has the same property. If M is even then $\varphi(x)/|x|^M$ is an analytic function, if M is odd then $\varphi(x)/|x|^{M+1}$ is an analytic function. Then it follows that (16) with $s_1=M$ or $s_1=M+1$, respectively, is satisfied. In order to check (17) we remark that

$$\varphi(x) = (-1)^M \int_{R_n} g(h)\, dh + \sum_{k=1}^{M} a_k (Fg)(vkx) \tag{132}$$

holds, where a_k are appropriate constants. In particular we have $\varphi(2^m x)=c$ if $|x|\geqslant\frac{1}{4}$ and m is large. This shows that (17) with $s_0=0$ is satisfied. Hence, the hypotheses of Theorem 1 and Corollary 5 (with $N=1$) are satisfied. Furthermore, $F^{-1}\varphi(t\cdot)\,Ff$ with $\varphi(x)$ from (129) yields (127) with vt instead of t. Then it follows from (63) with $N=1$ that (128) is, at least for small values of v, an equivalent quasi-norm in $F^s_{p,q}$, where we used that $\|F^{-1}\varphi_0 Ff \mid L_p\|$ can be replaced by $\|f \mid L_p\|$, cf. Step 2 of the proof of the proposition in 3.2. The corresponding assertion for the homogeneous spaces $\dot{F}^s_{p,q}$ reads as follows: (128) with $\int_0^\infty$ instead of $\int_0^v$ and without the term $\|f \mid L_p\|$ is an equivalent uasi-norm in $\dot{F}^s_{p,q}$. The assertion of the theorem for arbitrary $0<v<\infty$ and in particular for $v=\infty$ follows now from the proposition in 3.2 and Remark 22.

Remark 23. We complete the proofs of Theorem 6 and Corollary 14(ii), which also completes the proof of Corollary 13(ii). We restrict ourselves to (109) and (110) with $\int_{|h|\leqslant v}$ and $\int_0^v$ instead of $\int_{|h|\leqslant 1}$ and $\int_0^1$, respectively, where $0<v\leqslant\infty$. The proof for the remaining quasi-norms is the same. Let $0<p<\infty$, $0<q\leqslant\infty$ and $n/\min(p,q)<s<M$, where M is a natural number. As we remarked at the end of the proofs of Theorem 6 and Corollary 14 the modified quasi-norms (109) and (110) (with v instead of 1) can be estimated from above by $c\,\|f \mid F^s_{p,q}\|$. In order to prove the reverse inequality we use the quasi-norm from (128). We have

$$|(K^M(g,t)f)(x)| \leqslant c \sum_{l=0}^{\infty} 2^{-rl} \sup_{0<|h|\leqslant t2^l} |(\Delta_h^M f)(x)|, \tag{133}$$

where $r>0$ is at our disposal. Then (128) with $v=\infty$ yields

$$\|f \mid F^s_{p,q}\| \leqslant c\,\|f \mid L_p\| + c \sum_{l=0}^{\infty} 2^{-r'l} \times \left\| \left(\int_0^\infty t^{-sq} \sup_{0<|h|\leqslant t2^l} |(\Delta_h^M f)(x)|^q \frac{dt}{t} \right)^{1/q} \Bigg| L_p \right\|, \tag{134}$$

where $r'>0$ is at our disposal. We substitute $\tau = t2^l$ in the respective terms on the right-hand side of (134). Then we have an additional factor 2^{ls}. We choose $r'>s$. It follows that $\|f \mid F^s_{p,q}\|$ can be estimated from above by the quasi-norm in (110) with $\int_0^\infty$ instead of $\int_0^1$, and the proof is complete as far as this special case is concerned. Next we extend the proof to the quasi-norm in (110) with $\int_0^v$ instead of $\int_0^1$, where $0<v<\infty$. Let $n/\min(p,q)<\tilde{s}<s$. We have

$$\begin{aligned} &\left\| \left(\int_v^\infty t^{-sq} \sup_{0<|h|\leqslant t} \left| (\Delta_h^M f)(\cdot) \right|^q \frac{dt}{t} \right)^{1/q} \Bigg| L_p \right\| \\ &\leqslant c \left\| \left(\int_v^\infty t^{-\tilde{s}q} \sup_{0<|h|\leqslant t} |(\Delta_h^M f)(\cdot)|^q \frac{dt}{t} \right)^{1/q} \Bigg| L_p \right\| \\ &\leqslant c\,\|f \mid F^{\tilde{s}}_{p,q}\| \leqslant c_\varepsilon \|f \mid L_p\| + \varepsilon\,\|f \mid F^s_{p,q}\|, \end{aligned} \tag{135}$$

where $\varepsilon > 0$ is at our disposal. The last estimate in (135) follows from (120) with $\sum_{j=0}^{\infty}$ instead of $\sum_{j=-\infty}^{\infty}$ and from a corresponding counterpart of (124). The above proven assertion that (110) with $\int_0^\infty$ instead of $\int_0^1$ is an equivalent quasi-norm in $F^s_{p,q}$ and (135) prove the correspondig assertion with $\int_0^\nu$ instead of $\int_0^1$ in (110). It remains to show that $\|f \mid F^s_{p,q}\|$ can be estimated from above by the quasi-norm in (109) with $\int_{|h|<\nu}$ instead of $\int_{|h|\leqslant 1}$. We may assume that $\nu = \infty$ because the cases with $\nu < \infty$ can be treated afterwards on the basis of (135). We use again (127) and (128) with $\nu = \infty$. Let $1 \leqslant q \leqslant \infty$. By Hölder's inequality we have (modification if $q = \infty$)

$$|(K^M(g, t) f)(x)|^q \leqslant c \int_{R_n} g(h)\, |(\Delta^M_{th} f)(x)|^q\, dh$$
$$= ct^{-n} \int_{R_n} g\left(\frac{h}{t}\right) |(\Delta^M_h f)(x)|^q\, dh \tag{136}$$

and

$$\int_0^\infty t^{-sq}\, |(K^M(g, t) f)(x)|^q \frac{dt}{t}$$
$$\leqslant c \int_{R_n} |h|^{-sq}\, |(\Delta^M_h f)(x)|^q \frac{dh}{|h|^n}. \tag{137}$$

We put (137) in (128) and obtain the desired estimate in the case $1 \leqslant q \leqslant \infty$. Let $0 < q < 1$. Then we modify (133) by

$$|(K^M(g, t) f)(x)|$$
$$\leqslant c \sum_{l=0}^{\infty} 2^{-rl} \sup_{0<|h|\leqslant t2^l} |(\Delta^M_h f)(x)|^{1-q} \int_{R_n} \tilde{g}(h)\, |(\Delta^M_{th} f)(x)|^q\, dh, \tag{138}$$

where $\tilde{g}(h)$ is a non-negative function on R_n with $\sup_{h \in R_n} |h|^N \tilde{g}(h) < \infty$ for any natural number N. (138) remains valid if we take the qth power term by term. Afterwards we multiply with t^{-sq-1} and integrate over $0 < t < \infty$. Then we apply Hölder's inequality, based on $q + (1-q) = 1$, and obtain

$$\int_0^\infty t^{-sq}\, |(K^M(g, t) f)(x)|^q \frac{dt}{t}$$
$$\leqslant c \sum_{l=0}^{\infty} 2^{-rlq} \left(\int_0^\infty t^{-sq} \sup_{0<|h|\leqslant t2^l} |(\Delta^M_h f)(x)|^q \frac{dt}{t} \right)^{1-q}$$
$$\times \int_0^\infty t^{-sq} \int_{R_n} \tilde{g}(h)\, |(\Delta^M_{th} f)(x)|^q\, dh \frac{dt}{t} \Big)^q. \tag{139}$$

The second factors are independent of l and they can be estimated in the same way as in (136), (137). The first factors can be treated as above, i.e., we substitute $\tau = t2^l$ and choose $r > l$. Now we apply the L_p-quasi-norm and we use again Hölder's inequality with respect to $q + (1-q) = 1$. Because both (128) and (110) with $\int_0^\infty$ instead of $\int_0^1$ are equivalent quasi-norms in $F^s_{p,q}$ we obtain finally

$$\|f \mid F^s_{p,q}\| \leqslant c\, \|f \mid F^s_{p,q}\|^{1-q} \times \left(\|f \mid L_p\| + \left\| \left(\int_{R_n} |h|^{-sq}\, |(\Delta_h^M f)(\cdot)|^q \frac{dh}{|h|^n} \right)^{1/q} \Bigg| L_p \right\| \right)^q. \tag{140}$$

This yields the desired estimate in the case $0 < q < 1$.

Corollary 14 and the underlying Theorems 5 and 6 give the possibility to establish a Localization Principle which is useful, e.g., in order to study differential equations. We describe an example. Let $0 < p < \infty$, $0 < q \leqslant \infty$, and $n/\min(p, q) < s < M$ where M is a natural number. Let $\varepsilon > 0$ and let

$$F(x) = \left(\int_{|h| \leqslant \varepsilon} |h|^{-sq}\, |(\Delta_h^M f)(x)|^q \frac{dh}{|h|^n} \right)^{1/q}, \qquad x \in R_n. \tag{141}$$

Then $\|f \mid L_p\| + \|F \mid L_p\|$ is an equivalent quasi-norm in $F^s_{p,q}$, cf. (109) and Corollary 14(ii). However in order to calculate $F(x)$ one needs only the values of $f(y)$ with $|y - x| \leqslant \varepsilon M$, where one may choose $\varepsilon > 0$ as small as one wants. One has a similar assertion for the spaces $B^s_{p,q}$ provided that $0 < p \leqslant \infty$, $0 < q \leqslant \infty$ and $s > \tilde{\sigma}_p$; cf., e.g., Corollary 14(i) and (101) with $\int_{|h| \leqslant \varepsilon}$ instead of $\int_{|h| \leqslant 1}$. The question arises whether one can find for all spaces $B^s_{p,q}$ and $F^s_{p,q}$ without any restriction for s equivalent quasi-norms which exhibit this property, which we call the Localization Principle. In order to get an affirmative answer we introduce new weighted means of differences. Let $\psi \in S$ with

$$\operatorname{supp} \psi \subset \{ y \mid |y| \leqslant 1 \} \qquad \text{and} \qquad (F\psi)(0) \neq 0. \tag{142}$$

Let $t > 0$ and let M be a natural number. Then we put

$$[L(\psi, t) f](x) = \int_{R_n} \psi(y) f(x - ty)\, dy, \qquad x \in R_n \tag{143}$$

and

$$[L^M(\psi, t) f](x) = \int_{R_n} \int_{|h| \leqslant 1} \psi(y) (\Delta_{th}^M f)(x - ty)\, dh\, dy, \tag{144}$$

where the inner integral is taken with respect to $\{h \mid |h| \leqslant 1\}$ and the outer integral is taken with respect to $y \in R_n$. The integral over h is an (unweighted) ball mean of differences which is of the same structure as (127). In contrast to (127) we have now an additional weighted ball mean with respect to y; cf. (142). We use (143) and (144) for all $f \in S'$ which makes sense because $\psi \in S$. Recall that $\tilde{\sigma}_p$ has been defined in (12).

THEOREM 8. *Let* $0<q\leqslant\infty$ *and* $-\infty<s<\infty$. *Let* M *be a natural number with* $M>s$ *and* $M>\tilde{\sigma}_p$. *Let* $L(\psi, t)$ *and* $L^M(\psi, t)$ *be the means from* (143) *and* (144), *respectively.*

(i) *Let* $0<p\leqslant\infty$. *There exists a positive number* ε_0 *such that for all* ε *with* $0<\varepsilon<\varepsilon_0$,

$$\|L(\psi, \varepsilon) f \mid L_p\| + \left(\int_0^\varepsilon t^{-sq} \, \|L^M(\psi, t) f \mid L_p\|^q \frac{dt}{t}\right)^{1/q} \tag{145}$$

(*modification if* $q=\infty$) *is an equivalent quasi-norm in* $B^s_{p,q}$.

(ii) *Let* $0<p<\infty$. *There exists a positive number* ε_0 *such that for all* ε *with* $0<\varepsilon<\varepsilon_0$,

$$\|L(\psi, \varepsilon) f \mid L_p\| + \left\| \left(\int_0^\varepsilon t^{-sq} \, |(L^M(\psi, t) f)(\cdot)|^q \frac{dt}{t}\right)^{1/q} \Big| L_p \right\| \tag{146}$$

(*modification if* $q=\infty$) *is an equivalent quasi-norm in* $F^s_{p,q}$.

Proof. In order to prove part (ii) we use Theorem 1 with

$$\varphi_0(x) = (F\psi)(\varepsilon x) \qquad \text{and} \qquad \varphi(x) = (F\psi)(\varepsilon x) \int_{|h| \leqslant 1} (e^{i\varepsilon hx} - 1)^M \, dh, \tag{147}$$

with $\varepsilon > 0$, where hx stands for the scalar product of $h \in R_n$ and $x \in R_n$. If $\varepsilon > 0$ is small then both (14) and (15) are satisfied. Furthermore, $\varphi(x)/|x|^M$ if M is even, and $\varphi(x)/|x|^{M+1}$ if M is odd, are analytic functions. This shows that (16) with $s_1 = M$ (resp. $s_1 = M+1$) is satisfied. Furthermore, (17′) and (18′) from Corollary 3 are satisfied for any number s_0. Hence we can apply Theorem 1 where we now prefer the version (63) with $N=1$. Recall $t^{-n}(F\psi(t^{-1}\cdot))(x) = (F\psi)(tx)$. Then (147), (143), and (144) yield (146). Part (i) follows in the same way where one has to use Theorem 3 instead of Theorem 1.

Remark 24. Let $x \in R_n$ and $0<t<\varepsilon$. In order to calculate the values of the means in (143) and (144) at the point x one needs only the values of $f(z)$ with $|z-x| < \varepsilon(M+1)$. This makes sense for any $f \in S'$. This shows that all spaces $B^s_{p,q}$ with $0<p\leqslant\infty$, $0<q\leqslant\infty$, $s \in R_1$, and all spaces $F^s_{p,q}$

with $0<p<\infty$, $0<q\leqslant\infty$, and $s\in R_1$ satisfy the Localization Principle which we described in front of Theorem 8.

COROLLARY 15. *Let $0<q\leqslant\infty$ and $-\infty<s<\infty$. Let M be a natural number with $M>s$ and $M>\tilde{\sigma}_p$. Let $L(\psi, t)$ be the mean from* (143).

(i) *Let $0<p\leqslant\infty$. There exists a positive number ε_0 such that for all ε with $0<\varepsilon<\varepsilon_0$,*

$$\|L(\psi,\varepsilon)f \mid L_p\| + \sum_{|\alpha|=M}\left(\int_0^\varepsilon t^{(M-s)q}\,\|L(\psi,t)\,D^\alpha f \mid L_p\|^q\frac{dt}{t}\right)^{1/q} \tag{148}$$

(*modification if $q=\infty$*) *is an equivalent quasi-norm in $B^s_{p,q}$.*

(ii) *Let $0<p<\infty$. There exists a positive number ε_0 such that for all ε with $0<\varepsilon<\varepsilon_0$,*

$$\|L(\psi,\varepsilon)f \mid L_p\| + \sum_{|\alpha|=M}\left\|\left(\int_0^\varepsilon t^{(M-s)q}\,|(L(\psi,t)\,D^\alpha f)(\cdot)|^q\frac{dt}{t}\right)^{1/q}\,\middle|\, L_p\right\| \tag{149}$$

(*modification if $q=\infty$*) *is an equivalent quasi-norm in $F^s_{p,q}$.*

Proof. In order to prove part (ii) we use Corollary 5 with

$$\varphi_0(x)=(F\psi)(\varepsilon x) \qquad \text{and} \qquad \varphi^{(\alpha)}(x)=x^\alpha(F\psi)(\varepsilon x),\quad |\alpha|=M, \tag{150}$$

where $\varepsilon>0$, $x\in R_n$, and $x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$. As in the proof of Theorem 8 the function φ_0 satisfies (14) and (18′) from Corollary 3 for any s_0 provided that $\varepsilon>0$ is small. For small $\varepsilon>0$ we have also (62) and (17′) for any function $\varphi^{(\alpha)}$. As far as (16) or (16′) with $\varphi^{(\alpha)}$ instead of φ is concerned we refer to the modifications indicated at the end of Remark 19. This means in our case that we can replace $\varphi(z)\,h(z)/|z|^{s_1}$ in (16) by $\varphi^{(\alpha)}(z)\,h(z)/z^\alpha$ provided $|\alpha|=M=s_1>s$ and $s_1>\tilde{\sigma}_p$. Hence the hypotheses of Corollary 5 are satisfied. In the same way as in the proof of Theorem 8 part (ii) follows from (63). Remark 17 proves part (i).

Remark 25. Again $L(\psi, t)D^\alpha f$ must be interpreted in the sense of distributions. Furthermore the Localization Principle can also be established on the basis of the above corollary.

3.4. *Characterizations via Harmonic Functions and Temperatures*

It is well known that function spaces of Besov–Hardy type on R_n can be characterized as traces of harmonic functions or of temperatures in $R^+_{n+1}=\{(x,t)\mid x\in R_n,\ t\geqslant 0\}$ on the hyperplane $t=0$ which is identified with R_n. In our context this means that we have to choose the fuction $\varphi(x)$ from Theorem 1 and the subsequent theorems and corollaries as $e^{-|x|}$ (in the case of harmonic functions) and as $e^{-|x|^2}$ (in the case of temperatures) where we must multiply these functions with some powers of $|x|$. Recall that

$$(F^{-1}\varphi(t\cdot)\,Ff)(x)=c\int_{R_n}(F^{-1}\varphi(t\cdot))(x-y)f(y)\,dy$$
$$=ct^{-n}\int_{R_n}(F^{-1}\varphi)\left(\frac{x-y}{t}\right)f(y)\,dy. \qquad (151)$$

If we choose $\varphi(x)=e^{-|x|}$ then we have

$$(F^{-1}e^{-|\xi|})(x)=\frac{c}{(1+|x|^2)^{(n+1)/2}}$$

and (151) yields the Cauchy–Poisson semi-group $P(t)f$ of harmonic functions in R^+_{n+1} given by

$$[P(t)f](x)=c\int_{R_n}\frac{t}{(|x-y|^2+t^2)^{(n+1)/2}}f(y)\,dy, \qquad x\in R_n, \quad t>0. \qquad (152)$$

If we choose $\varphi(x)=e^{-|x|^2}$ then we have $F(e^{-|\xi|^2/2})(x)=e^{-|x|^2/2}$ and (151) with $\varphi(\sqrt{t}\,\cdot)$ instead of $\varphi(t\cdot)$ yields the Gauss–Weierstrass semi-group $W(t)f$ of temperatures in R^+_{n+1}, given by

$$[W(t)f](x)=ct^{-n/2}\int_{R_n}e^{-|x-y|^2/4t}f(y)\,dy, \qquad x\in R_n, \quad t>0. \quad (153)$$

Recall that $u(x,t)=[W(t)f](x)$ satisfies the heat equation in $\{(x,t)\mid x\in R_n,\ t>0\}$. As far as the formal aspects of these two semi-groups are concerned we refer to [20, 2.5.2, 2.5.3]. If $f\in S'$ then (153) makes sense. In the case of (152) one must interprete this expression in a sense of a limiting procedure. Recall that $\tilde{\sigma}_p$ has been defined in (12).

THEOREM 9. *Let $0<q\leqslant\infty$ and $-\infty<s<\infty$. Let $\varphi_0\in S$ with $\varphi_0(0)\neq 0$.*

(i) *Let $0<p\leqslant\infty$. Let k and m be non-negative integers with $k>\max(s,\tilde{\sigma}_p)+\tilde{\sigma}_p$ and $2m>s$. Then*

$$\|F^{-1}\varphi_0 Ff\mid L_p\|+\left(\int_0^1 t^{(k-s)q}\left\|\frac{\partial^k P(t)f}{\partial t^k}\,\middle|\,L_p\right\|^q\frac{dt}{t}\right)^{1/q} \qquad (154)$$

and

$$\|F^{-1}\varphi_0 Ff\mid L_p\|+\left(\int_0^1 t^{(m-s/2)q}\left\|\frac{\partial^m W(t)f}{\partial t^m}\,\middle|\,L_p\right\|^q\frac{dt}{t}\right)^{1/q} \qquad (155)$$

(modification if $q=\infty$) are equivalent quasi-norms in $B^s_{p,q}$. If $s>\tilde{\sigma}_p$ then $\|F^{-1}\varphi_0 Ff \mid L_p\|$ in (154) and (155) can be replaced by $\|f \mid L_p\|$.

(ii) *Let $0<p<\infty$. Let k and m be non-negative integers with $k>\max(s, \tilde{\sigma}_p)+n/\min(p, q)$ and $2m>s$. Then*

$$\|F^{-1}\varphi_0 Ff \mid L_p\| + \left\| \left(\int_0^1 t^{(k-s)q} \left| \frac{\partial^k P(t) f}{\partial t^k}(\cdot) \right|^q \frac{dt}{t} \right)^{1/q} \,\Big|\, L_p \right\| \tag{156}$$

and

$$\|F^{-1}\varphi_0 Ff \mid L_p\| + \left\| \left(\int_0^1 t^{(m-(s/2))q} \left| \frac{\partial^m W(t) f}{\partial t^m}(\cdot) \right|^q \frac{dt}{t} \right)^{1/q} \,\Big|\, L_p \right\| \tag{157}$$

(modification if $q=\infty$) are equivalent quasi-norms in $F^s_{p,q}$. If $s>\tilde{\sigma}_p$ then $\|F^{-1}\varphi_0 Ff \mid L_p\|$ in (156) and (157) can be replaced by $\|f \mid L_p\|$.

Proof. Step 1. We prove part (ii). We use Theorem 1 with the above function φ_0 and $\varphi(x)=|x|^k e^{-|x|}$. The above assumption $\varphi_0(0)\neq 0$ and $\varphi(x)\neq 0$ for all $x\neq 0$ cover (14) and (15). Furthermore, (17′) and (18′) from Corollary 3 are satisfied for any s_0. Finally, (16′) can be reduced to the question whether $e^{-|x|}|x|^{k-s_1}$ with $s_1>\max(s, \tilde{\sigma}_p)$ belongs to H^σ_2 with $\sigma>(n/2)+(n/\min(p, q))$. By Remark 6 this property holds if $k-s_1+(n/2)>\sigma$. This is satisfied by the above assumption for k. Hence we can apply Theorem 1 and (63) with $N=1$. We have

$$\begin{aligned} F^{-1}\varphi(t\cdot)\, Ff &= t^k F^{-1}\, |y|^k\, e^{-t|y|} Ff \\ &= t^k \frac{\partial^k}{\partial t^k} F^{-1} e^{-t|y|} Ff = t^k \frac{\partial^k}{\partial t^k} P(t) f; \end{aligned} \tag{158}$$

cf. (152), which proves that (156) is an equivalent quasi-norm in $F^s_{p,q}$. In order to prove the corresponding assertion for the expression in (157) we use again Theorem 1 with φ_0 and $\varphi(x)=|x|^{2m} e^{-|x|^2}$. Then all conditions, including (16′) with $s_1=2m>s$ are satisfied, and the counterpart of (158) reads as

$$\begin{aligned} F^{-1}\varphi(\sqrt{t}\cdot)\, Ff &= t^m F^{-1}\, |y|^{2m}\, e^{-t|y|^2} Ff \\ &= t^m \frac{\partial^m}{\partial t^m} F^{-1} e^{-t|y|^2} Ff = t^m \frac{\partial^m}{\partial t^m} W(t) f; \end{aligned} \tag{159}$$

cf. (153). Then it follows that (157) is an equivalent quasi-norm in $F^s_{p,q}$ where one has to take into consideration that we substituted t by $\sqrt{t}$. Finally let $s>\tilde{\sigma}_p$. Then we have (123) and (124) which prove that we can replace $\|F^{-1}\varphi_0 Ff \mid L_p\|$ in (156) and (157) by $\|f \mid L_p\|$.

Step 2. Part (i) can be proved in the same way. We use Theorem 3 and Corollary 9 instead of Theorem 1 and Corollary 3, respectively.

Remark 26. A reasonable choice of φ_0, at least in (155) and (157), is given by $\varphi_0(x) = e^{-c|x|^2}$ with $c > 0$. In other words, one can replace $\|F^{-1}\varphi_0 Ff \mid L_p\|$ in (154)–(157) by $\|W(1)f \mid L_p\|$. On the other hand, $\|F^{-1}\varphi_0 Ff \mid L_p\|$ cannot be replaced by $\|P(1)f \mid L_p\|$, in general. A more detailed study of this question can be based on (152), but we shall not go into detail.

COROLLARY 16. *Let* $0 < q \leqslant \infty$ *and* $-\infty < s < \infty$.

(i) *Let* $0 < p \leqslant \infty$. *Let* k *and* m *be non-negative integers with* $k > s + \tilde{\sigma}_p$ *and* $2m > s$. *Then*

$$\left(\int_0^\infty t^{(k-s)q} \left\| \frac{\partial^k P(t) f}{\partial t^k} \;\middle|\; L_p \right\|^q \frac{dt}{t}\right)^{1/q} \tag{160}$$

and

$$\left(\int_0^\infty t^{(m-s/2)q} \left\| \frac{\partial^m W(t) f}{\partial t^m} \;\middle|\; L_p \right\|^q \frac{dt}{t}\right)^{1/q} \tag{161}$$

(*modification if* $q = \infty$) *are equivalent quasi-norms in* $\dot{B}^s_{p,q}$.

(ii) *Let* $0 < p < \infty$. *Let* k *and* m *be non-negative integers with* $k > s + (n/\min(p, q))$ *and* $2m > s$, *then*

$$\left\| \left(\int_0^\infty t^{(k-s)q} \left| \frac{\partial^k P(t) f}{\partial t^k}(\cdot) \right|^q \frac{dt}{t}\right)^{1/q} \;\middle|\; L_p \right\| \tag{162}$$

and

$$\left\| \left(\int_0^\infty t^{(m-(s/2))q} \left| \frac{\partial^m W(t) f}{\partial t^m}(\cdot) \right|^q \frac{dt}{t}\right)^{1/q} \;\middle|\; L_p \right\| \tag{163}$$

(*modification if* $q = \infty$) *are equivalent quasi-norms in* $\dot{F}^s_{p,q}$.

Proof. Terms (160)–(163) are the homogeneous counterparts of (154)–(157), respectively. Instead of Theorems 1 and 3 we have to use Corollaries 2(i) and 8(i), respectively. The counterpart of (63) has been mentioned in Remark 18.

3.5. *Comments*

The results of this paper generalize, modify, and, in particular, unify, corresponding assertions from [22] for the spaces $B^s_{p,q}$ and $F^s_{p,q}$. As far as the general history of these spaces is concerned we refer to [22, 2.3.5]. In

[22, 2.5] we gave several characterizations of the spaces $B^s_{p,q}$ and $F^s_{p,q}$ in terms of differences and derivatives of functions and related ball means. However, we used rather specific methods. A first step in direction of a more unified approach was done in [23] and the Subsections 2.5.15–2.5.17 of the recent Russian edition of [22]. The present paper may be considered as the continuation of this way. The main new ingredients are the conditions of Tauberian type, cf. for instance (14), (15), or (54), and the rather careful description of the assumptions of the underlying functions in the sense of (16)–(18), etc. This is the basis for a unified approach to the apparently rather different types of equivalent quasi-norms in the treated spaces presented in this paper. As far as the use of Tauberian conditions is concerned we gave some references in Remark 10. Concrete characterizations of function spaces via derivatives and differences of functions, and harmonic or thermic extensions are known. Characterizations of the classical Besov–Sobolev spaces based on derivatives and differences may be found in [11] and [20], including many historical remarks and references. The use of semi-groups of operators in Banach spaces in connection with the classical Besov–Sobolev spaces and related problems in approximation theory has been studied in [4] and [20]. This covers not only characterizations via derivatives and differences (translation group) but also characterizations via harmonic and thermic extensions (Cauchy–Poisson, and Gauss–Weierstrass semi-groups, respectively). However, the semi-group approach is not effective for the spaces $F^s_{p,q}$ (even not if $p \geqslant 1$ and $q \geqslant 1$) and it breaks down if $p<1$ in $B^s_{p,q}$ or $F^s_{p,q}$. As far as the spaces $F^s_{p,q}$ with $s>0$, $1<p<\infty$, $1<q<\infty$ are concerned a detailed study of equivalent norms involving derivatives and differences of functions, as well as harmonic and thermic extensions has been given by Kaljabin [8, 9]. Finally we refer to the papers [2, 3] by Bui, where characterizations of spaces with $p<1$ (and with weights of Muckenhoupt type in the case of the latter paper) in terms of thermic and harmonic extensions are given.

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